The Transitive Graphs with at Most 26 Vertices

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Abstract. We complete the construction of all the simple graphs with at most 26 vertices and transitive automorphism group. The transitive graphs with up to 19 vertices were earlier constructed by McKay, and the transitive graphs with 24 vertices by Praeger and Royle. Although most of the construction was done by computer, a substantial preparation was necessary. Some of this theory may be on independent interest.

1. Introduction

Let G be a finite simple graph with automorphism group Aut(G). If Aut(G) acts transitively on $V(G)$, then we say that G is *transitive*. The aim of this paper is to describe the methods by which the complete set of transitive graphs of order at most 26 has been generated.

The transitive graphs on a prime number p of vertices are the graphs whose automorphism groups contain a p-cycle. The isomorphism classes were determined by Elspas and Turner [5].

For the case when the number of vertices is $2p$, p prime, Alspach and Sutcliffe [1] described a particular family of transitive graphs and conjectured that there were no others. The truth of their conjecture follows from results of Masrušič [15] in conjunction with a corollary of the classification of the finite simple groups (that there are no simply-transitive primitive permutation groups of degree $2p$ for $p \neq 5$).

For other orders, few general results are known. H.P. Yap made the first significant attempt at a catalogue; he found all the transitive graphs up to 11 vertices, and many classes of them on 12 vertices. A complete list of transitive graphs up to 19 vertices was compiled by McKay [18] and published in [17]. The method of construction was not described in [17], however; that will be the subject of our Sections 2 and 3. The transitive graphs on 20–23 vertices were found by McKay

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and Royle [24]; we will describe this construction in Section 4. Section 4 also describes, for the first time, the construction of the transitive graphs on 25 or 26 vertices. Finally, the transitive graphs on 24 vertices were found by Royle and Praeger [24, 25]; we will not repeat this construction here.

A few related compilations can be mentioned here. The circulant graphs (those on n vertices whose automorphism group contains an n-cycle) were found up to order 37 by the first author in 1977 (unpublished). Graphs of order up to 11 with isomorphic vertex neighbourhoods were found by J. Hall [11]. D.H. Rees [23] determined all the cubic symmetric graphs of order up to 40 (G is symmetric if Aut(G) acts transitively on the directed edges of G); more extensive classifications or compilations of cubic transitive graphs were performed by Coxeter, Frucht and Powers [4] and Lorimer [12, 14]. A classification of symmetric graphs of prime degree was made by Lorimer [13]. The transitive planar graphs were completely classified by Fleischner and Imrich [6]. The complete list of Cayley graphs to 23 vertices was constructed in 1977 by the first author (unpublished) and to 31 vertices in 1986 by the second author [24]. Finally, R. Mathon [16] found all transitive self-complementary graphs with less than 50 vertices.

2. Theoretical Background

We will assume that the reader is conversant with the elementary terminology of graph theory and group theory. Only simple graphs will be considered. We will denote an edge $\{x, y\}$ of a graph as xy for brevity. $E(G)$ is the edge-set of G and $\overline{G}$ is the complement of G. The set of neighbours of v in G will be denoted by $N(v, G)$, and $V(G)\backslash(\{v\} \cup N(v, G))$ will be denoted by $\overline{N}(v, G)$.

Suppose that Λ is a set of permutations (not necessarily a group) acting on a set V. The *support* supp(Λ) of Λ is the set of elements of V moved by some element of Λ, while the *fixed-point set* fix(Λ) of Λ is the set of elements of V fixed by every element of Λ. Obviously, supp(Λ) $\cup$ fix(Λ) $= V$.

If G is any graph, then the *switching graph* of G, denoted Sw(G), has $V(\mathrm{Sw}(G)) = V(G) \times \{0, 1\}$ and $E(\mathrm{Sw}(G)) = \{(x, i)(y, j) \mid i = j \text{ and } xy \in E(G), \text{ or } i \neq j \text{ and } xy \in E(\overline{G})\}$. Switching graphs have relevance to the *switching classes* of [26]; in particular, two graphs are in the same switching class if and only if their switching graphs are isomorphic [8].

If G and H are graphs, the *lexicographic product* $G[H]$ has $V(G[H]) = V(G) \times V(H)$ and $E(G[H]) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(G) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(H)\}$. We will say that G is a *non-trivial lexicographic product* (NTLP) if $G = H[J]$ for some graphs H and J with at least two vertices. The importance of NTLPs to us comes from the following lemma. A subset $W \subseteq V(G)$ is called *externally related* (ER) in G if each pair of vertices in W are adjacent to exactly the same vertices in $V(G)\backslash W$. W is a *non-trivial* ER subset if $2 \leq |W| \leq V(G) - 1$.

Theorem 2.1. *Let G be a transitive graph which is neither empty nor complete. Then the following are equivalent.*

(a) *G is an NTLP.*

(b) *$G = H[J]$ for some transitive graphs H and J with at least two vertices.*

(c) *G has a non-trivial ER subset.*

(d) Aut(G) *has a non-trivial ER block.*

(e) Aut(G) *has an intransitive subgroup with exactly one orbit of length greater than one.*

Proof: Obviously, (b)⇒(a)⇒(c) and (d)⇒(e)⇒(c), so that it will suffice to prove that (c)⇒(d)⇒(b).

Suppose that condition (c) is satisfied. Let W be a non-trivial ER subset of the least possible size. If Aut(G) constains no transpositions, then $|W| \geq 3$. Now, for each $\gamma \in$ Aut(G), if $W \cap W^\gamma \neq \emptyset$ then $W^\gamma = W$ since otherwise one of $W \cap W^\gamma$ and $W \backslash W^\gamma$ would be a non-trivial ER subset smaller than W. Suppose alternatively that Aut(G) contains a transposition $(x\ y)$. By replacing G by $\overline{G}$ if necessary, we have $N(x,G) = N(y,G)$. Then $\{v \in V(G) | N(v,G) = N(x,G)\}$ is a non-trivial ER block of Aut(G) or else G is empty.

Suppose that condition (d) is satisfied and let $B_1, B_2, \ldots, B_r$ be the corresponding complete block system. Since Aut(G) acts transitively on the blocks, each B_i is ER and induces an isomorphic subgraph of G. Thus, each distinct pair B_i and B_j are joined either by no edges of G or by all possible edges. Condition (b) is thus satisfied. ∎

The implications (a)⇔(e) were first proved by C. Godsil. As sample applications of Theorem 2.1, we have the following theorems.

Theorem 2.2. *Let G be a non-complete connected transitive graph. If $\overline{N}(v,G)$ is disconnected for some $v \in V(G)$, then G is an NTLP.*

Proof: By Gardiner [7] or Ashbacher [2], either $N(v,g) = N(w,G)$ for some $v \neq w$ (implying that $(v\ w) \in$ Aut(G)) or G has a non-trivial ER block. Theorem 2.1 applies immediately in either case. ∎

Theorem 2.3. *Let G be a connected non-complete transitive graph with odd order $n \geq 7$. If* Aut(G) *contains a non-trivial subgroup Λ which moves at most 7 vertices, then G is an NTLP.*

Proof: By considering all the possibilities for Λ, we see that Λ contains a subgroup satifying part (e) of Theorem 2.1 or else a subgroup of the form $\langle (a\ b)(c\ d) \rangle$ or $\langle (a\ b\ c)(d\ e\ f) \rangle$. In the latter case, consider all the possibilities for the subgraph induced by $\{a,b,c,d,e,f\}$; in every case we find that $(a\ b)(d\ e) \in \Lambda$. Now suppose $\mathcal{D}(G) \neq \emptyset$, where $\mathcal{D}(G)$ is the set of all elements of Aut(G) of the form $(a\ b)(c\ d)$. Since n is odd, and Aut(G) is transitive, there are distinct

$\gamma, \delta \in \mathcal{D}(G)$ such that $\text{supp}(\gamma) \cap \text{supp}(\delta) \neq \emptyset$. Now consider all the ways that γ and δ can overlap. In most cases, $\langle \gamma, \delta \rangle$ contains a subgraph satisfying part (e) of Theorem 2.1. [For example, take $\gamma = (a\ b)(c\ d)$ and $\lambda = (a\ e)(c\ f)$, where all these vertices are distinct. Then $\langle \gamma, \delta\gamma\delta \rangle$ contains exactly one orbit.] The only exception is when γ and δ overlap as do $(a\ b)(c\ d)$ and $(a\ e)(c\ f)$, so assume that all non-trivial overlaps between elements of $\mathcal{D}(G)$ have this form. Define a relation "$\sim$" on $V(G)$:

(i) $x \sim x$ for all x.

(ii) If $x \neq y$, then $x \sim y$ if and only if there are elements $\gamma = (x\ a)(y\ b)$ and $\delta = (x\ c)(y\ d)$ of $\mathcal{D}(G)$ such that $a \neq c$ and $b \neq d$.

It is easily seen that "$\sim$" is an equivalence relation with classes of size 2, contradicting the assumption that n is odd. ∎

If Λ_1 and Λ_2 are permutation groups acting on a set V, and $\text{supp}(\Lambda_1) \cap \text{supp}(\Lambda_2) = \emptyset$, then we will write $\Lambda_1 \oplus \Lambda_2$ for the group $\langle \Lambda_1, \Lambda_2 \rangle$. Clearly, $\Lambda_1 \oplus \Lambda_2$ is isomorphic as an abstract group to the direct product of Λ_1 and Λ_2, but the permutation representation is important to us here. If Λ is a non-trivial permutation group acting on V, then Λ has a unique representation

$$\Lambda = \Lambda^{(1)} \oplus \Lambda^{(2)} \oplus \cdots \oplus \Lambda^{(r)},$$

where the supports of the $\Lambda^{(i)}$ are non-empty and disjoint, and r is maximum. We will refer to the groups $\Lambda^{(i)}$ as the *fragments* of Λ.

For a group Λ and a prime p, let $\text{Syl}_p(\Lambda)$ be the set of all Sylow p-subgroups of Λ.

Lemma 2.4. *Let G be a transitive graph, and let $v \in V(G)$. Suppose that $P \in \text{Syl}_p(\Gamma_v)$, where $\Gamma = \text{Aut}(G)$, Γ_v denotes the point stabiliser of v in Γ, and p is a prime dividing $|\Gamma_v|$. Define a graph $H = H(G,P)$ as follows. $V(H)$ is the set of non-trivial orbits of P. Two distinct vertices of H are adjacent if and only if the corresponding orbits of P are joined by some edges of G but not completely joined in G. Then the supports of the fragments of P correspond to the components of H.*

Proof: Let $P^{(1)}, P^{(2)}, \ldots, P^{(r)}$ be the fragments of P. Since the action of P on orbits in different fragments is independent, fragments correspond to unions of orbits. Now suppose that the support of a fragment $P^{(i)}$ is $V_1 \cup V_2$, where V_1 and V_2 are disjoint non-empty sets of orbits each of which corresponds to a union of components of H. Then the restrictions $P|_{V_1}$ and $P|_{V_2}$ are each in Γ by the structure of G. However, $P|_{V_1} \oplus P|_{V_2}$ is not in $P^{(i)}$ since $P^{(i)}$ is a fragment. Thus $P^{(1)} \oplus \cdots \oplus P^{(i-1)} \oplus P|_{V_1} \oplus P|_{V_2} \oplus P^{(i+1)} \oplus \cdots \oplus P^{(r)}$ is a p-subgroup of Γ_W larger than P, contradicting the assumption that $P \in \text{Syl}_p(\Gamma_v)$. ∎

If $\Lambda \le \Phi \le \Gamma$ are groups, we say that Λ is *weakly-closed* in Φ with respect to Γ if, for each $\gamma \in \Gamma$, $\Lambda^\gamma \le \Phi$ if and only if $\Lambda^\gamma = \Lambda$.

Lemma 2.5. *Suppose that Γ is a group acting transitively on $V = \{1, 2, \ldots, n\}$, and let $1 < P \bullet \mathrm{Syl}_p(\Gamma_1)$ for some prime p. If $1 < \Lambda \le P$ and Λ is weakly closed in P with respect to Γ, then $|\mathrm{fix}(\Lambda)| \le n/2$.*

Proof: Suppose that $|\mathrm{fix}(\Lambda)| > n/2$. Let $\gamma \in \Gamma$ and $\Phi \bullet \langle \Lambda, \Lambda^\gamma \rangle$. Then $|\mathrm{fix}(\Phi)| \ge 1$, so that $\Phi \le \Gamma_x$ for some $x \in V$. By Sylow's Theorem, there are $Q \in \mathrm{Syl}_p(\Phi)$ and $\phi \in \Phi$ such that $\Lambda \le Q$ and $\Lambda^\gamma \le Q^\phi$. But then Λ^ϕ and Λ^γ are both in Q^ϕ and hence in any conjugate of P which contains Q^ϕ. Therefore $\Lambda^\phi = \Lambda^\gamma$ by the weak closure condition. But then $|\mathrm{fix}(\Lambda)| \le (n-1)/2$ by a result of C. Praeger [22], contradicting our assumption. ∎

Theorem 2.6. *Assume the definitions of Lemma 2.4. Suppose that some fragment Φ of P is uniquely identified amongst the fragments of P by the sizes of its orbits and that, for every $\gamma \bullet \Gamma$, $\Phi \le P$ only if the non-trivial orbits of Φ^γ are orbits of P. Then $|\mathrm{supp}(\Phi)| \ge n/2$.*

Proof: Let $\gamma \bullet \Gamma$. If $\Phi^\gamma \le P$ then, by assumption, $\mathrm{supp}(\Phi)$ is a union of orbits of P. Since $\mathrm{supp}(\Phi^\gamma)$ is a component of $H(G, P^\gamma)$, $\mathrm{supp}(\Phi^\gamma) = \mathrm{supp}(\Phi')$ for some fragments Φ' of P. But then $\Phi^\gamma \le \Phi'$ since $\Phi^\gamma \le P$ and so $\Phi^\gamma = \Phi'$, since both Φ^γ and Φ' are Sylow p-subgroups of the subgroup of Γ which fixes the orbits of Φ' setwise.

It follows that Φ is weakly closed in P with respect to Γ, so Lemma 2.5 applies. ∎

As an example of the use of Theorem 2.6, consider the automorphism group of a transitive graph with 15 vertices. A Sylow 2-subgroup cannot have the form $\langle (2\ 3)(4\ 5)(6\ 7), (8\ 9)(10\ 11)(12\ 13)(14\ 15) \rangle$, since the fragment $\langle (2\ 3)(4\ 5)(6\ 7) \rangle$ has a support which is too small.

Next we classify some types of subgroups of Γ_w where Γ is a group acting transitively on V, $w \bullet V(G)$ and p is a prime dividing $|\Gamma_w|$:

(a) Γ_w itself is a *type-1* subgroup.
(b) Any $\Lambda \in \mathrm{Syl}_p(\Gamma_w)$ is a *type-2* subgroup.
(c) The subgroup $\langle \mathrm{Syl}_p(\Gamma_w) \rangle$ generated by all the Sylow p-subgroups of Γ_w is a *type-3* subgroup.
(d) Suppose that $\Lambda \in \mathrm{Syl}_p(\Gamma_w)$ has $\mathrm{fix}(\Lambda) = \{w\}$. If Λ has an orbit X of size p and $|\Lambda_x| > 1$ for $x \in X$, then Λ_x is a *type-4* subgroup.

Theorem 2.7.

(a) *If Λ is a subgroup of Γ_w of type 1, 2, 3 or 4, then the normaliser $N_\Gamma(\Lambda)$ acts transitively on* $\mathrm{fix}(\Lambda)$.
(b) *If Λ is a subgroup of Γ_w of type 1 or 3, then* $\mathrm{fix}(\Lambda)$ *is a block of* Γ.

Proof: In (a), Λ is a conjugate in Γ_w to any of its conjugates in Γ which lie in Γ_w. (For type-4 subgroups, Lemma 7.4.7 of [**10**] is required.) We can thus apply Jordan's theorem (Theorem 3.5 of [**29**]). Claim (b) is an elementary exercise. ∎

We now turn to some applications of linear algebra. Let G be a transitive graph with $V(G) = \{1,2,\ldots,n\}$. Let A be the $(0,1)$ adjacency matrix of G. Suppose that $\Lambda \leq \mathrm{Aut}(G)$ and let $V_1, V_2, \ldots, V_m$ be the orbits of Λ in lexicographical order. The $m \times m$ matrix $Q(G,\Lambda) = (q_{ij})$ is defined by

$$q_{ij} = \sqrt{|V_j|/|V_i|}\, e_{ji},$$

where e_{ij} is the number of vertices in V_j to which each vertex in V_i is adjacent in G. It is not completely obvious, but true, that $Q(G,\Lambda)$ is symmetric.

For any real symmetric matrix M and real number λ, define $\mu_M(\lambda)$ to be zero if λ is not an eigenvalue of M and the multiplicity of λ as an eigenvalue of M otherwise.

Theorem 2.8. *Suppose that V_t contains a single vertex for some t. Let $Q = Q(G,\Lambda)$. For any real number λ, define $\rho(Q,\lambda,t)$ as follows.*

- (a) *If λ is not an eigenvalue of Q, $\rho(Q,\lambda,t) = 0$.*
- (b) *If λ is an eigenvalue of Q, let $x_1, x_2, \ldots, x_s$ be a complete orthonormal set of eigenvectors of Q for λ. Then define $\rho(Q,\lambda,t) = \sum_{i=1}^{s}(x_i)_t^2$, where the summand is the square of the t-th entry of x_i.*

Then $\mu_A(\lambda) = n\rho(Q,\lambda,t)$.

Proof: This is a special case of Theorem 3.4 of [**9**]. ∎

Corollary 2.9. *$\rho(Q,\lambda,t)$ is independent of t so long as $|V_t| = 1$.* ∎

Corollary 2.10. *$Q(G,\Lambda)$ and A have the same eigenvalues up to multiplicities.* ∎

We will also have use for the following facts about simple eigenvalues of A.

Theorem 2.11. *Let A be the $(0,1)$-adjacency matrix of a vertex transitive graph G of order n and degree k.*

- (a) *If λ is a simple eigenvalue of A, then λ is an integer of the form $k - 2\alpha$, for integer α.*
- (b) *n is even if A has at least two simple eigenvalues, and divisible by four if A has at least three simple eigenvalues.*

Proof: Part (a) was first proved by Petersdorff and Sachs [**21**]. A proof of part (b) can be found in [**9**]. ∎

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