

# THE SPECTRUM OF A GRAPH

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## ABSTRACT

We survey the results obtained by a large number of authors concerning the spectrum of a graph. The questions of characterisation by spectrum, cospectral graphs and information derived from the spectrum are discussed.

### 1: INTRODUCTION

Our aim here is to review a large number of papers which deal with the spectrum of a graph, bringing the results together in one place for easy and convenient access. Prior to this, two similar surveys have been undertaken by Cvetković [16] and Wilson [74]. We attempt to bring these surveys up to date.

Throughout we will consider simple graphs on a finite set of vertices. The adjacency matrix  $A(G)$  of a graph  $G$  on  $n$  vertices, is the  $n \times n$  matrix  $(a_{ij})$ , where  $a_{ij} = 1$  if vertex  $i$  is adjacent to vertex  $j$  and  $a_{ij} = 0$ , otherwise. By  $G(\lambda)$  we mean the characteristic polynomial of the graph  $G$ . This we define to be the characteristic polynomial,  $\det(\lambda I - A(G))$ . We note that some authors prefer  $\det(A(G) - \lambda I)$ , but that this clearly does not affect the results obtained. The eigenvalues of  $G$  are the eigenvalues of  $A(G)$ , and the set of all such eigenvalues is the spectrum of  $G$ . The spectral radius of  $G$ ,  $\text{spec. rad.}(G)$ , is the largest eigenvalue of  $G$ .

The first paper published in this area appears to have been that by Collatz and Sinogowitz [11] in 1957. But obviously work had been done in this area long before, as one author, Sinogowitz, had already been dead for 13 years when the paper appeared. Further, some of the results of [11] were reviewed in [10].

Collatz and Sinogowitz put  $G(\lambda) = \sum_{r=0}^n a_r \lambda^{n-r}$  and were able to obtain relations between some of the coefficients  $a_r$  and certain graphical properties. For instance, they proved

THEOREM 1.1.  $a_0 = 1$ ;  $a_1 = \text{tr}(A(G)) = 0$ ;  $a_2 = -|E(G)|$ ;  $a_3 = -2t$ , where  $t$  is the number of triangles in  $G$ ;  $a_4 = N(2P_2) - 2N(C_4)$ ;  $a_5 = 2N(P_2 \cup C_3) - 2N(C_5)$ , where  $N(H)$  is the number of copies of  $H$  in  $G$ , and  $\text{tr}(A(G))$  is the trace of the matrix  $A(G)$ . □

This result is generalised in [57] (see Theorem 5.1).

A related result of Collatz and Sinogowitz is

THEOREM 1.2. If  $G$  is connected bipartite, then  $a_i = 0$  for all odd  $i$ . □

(This condition was shown to be sufficient by Gantmacher in [28]. He also proved that for  $G$  connected,  $G$  is bipartite if and only if  $-\text{spec. rad.}(G)$  is an eigenvalue of  $G$ .)

In addition, Collatz and Sinogowitz determined the eigenvalues of  $P_n$ ,  $C_n$ ,  $K_n$ , all the graphs with fewer than 6 vertices, and all the trees with fewer than 9 vertices.

They also considered the spectral radius of a connected graph  $G$ , and achieved the following result.

THEOREM 1.3. (a) if  $G^*$  is obtained from  $G$  by adding a new edge, then  $\text{spec. rad.}(G) < \text{spec. rad.}(G^*)$ .

(b) Average degree of  $G \leq \text{spec. rad.}(G) \leq \text{maximum degree of } G$ .

(c)  $2 \cos \frac{\pi}{n+1} \leq \text{spec. rad.}(G) \leq n-1$ . □

Many of the basic properties of the spectrum of  $G$  can be derived from elementary matrix theory. A number of these results are listed in [74] and we omit them here.

At first sight one might hope that the spectrum of a graph might somehow characterise it, but it is very easy to find graphs with the same spectrum. Examples were already available in [11]. Consequently we say that two non-isomorphic graphs are cospectral if they have the same characteristic polynomial. The smallest pair of cospectral graphs are  $K_{1,4}$  and  $C_4 \cup K_1$ , a fact that seems to have

been discovered a number of times. Collatz and Sinogowitz also knew of a pair of cospectral trees on 8 vertices. We will pursue this line of investigation in Section 4.

The eigenvalues or characteristic polynomials for a number of graphs are known and are to be found in [5], [11]\*, [13], [45], [53], [63], [71].

## 2: BASIC RESULTS

There are a number of straightforward results that follow directly from matrix theory and which we omit here due to lack of space. For instance, the eigenvalues of any graph are real and the spectrum of a disconnected graph is the union of the spectra of its components. Such results are readily available in [16] and [74].

We first give some results on methods of combining two graphs and the way in which the initial and final spectra are related.

THEOREM 2.1. If  $G + H$  is the join of the graphs  $G, H$  on  $m, n$  vertices respectively, then

$$(G + H)(\lambda) = (-1)^n G(\lambda) \bar{H}(-\lambda - 1) + (-1)^m H(\lambda) \bar{G}(-\lambda - 1) \\ - (-1)^{m+n} \bar{G}(-\lambda - 1) \bar{H}(-\lambda - 1),$$

where  $\bar{G}, \bar{H}$  are the complements of  $G, H$ , respectively.

Proof: See [16]. When  $G$  and  $H$  are regular, the above result simplifies to that obtained by Finck and Grohmann [26]. Incidentally, if  $G$  is regular of degree  $r$  with  $n$  vertices,

$$(-1)^n \frac{\bar{G}(-\lambda - 1)}{(\lambda + r - n)} = \frac{G(\lambda)}{(\lambda - r)}. \quad \square$$

THEOREM 2.2. If the eigenvalues of  $G, H$  are  $\lambda_i, \mu_j$ , respectively, then

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\* There are some errors here.

$$(G \times H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i - \mu_j),$$

$$(G \wedge H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i \mu_j),$$

and

$$(G * H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i \mu_j - \lambda_i - \mu_j).$$

Here  $G \times H$  represents the cartesian product,  $G \wedge H$  the conjunction and  $G * H$  the strong product.

Proof: See [63]. The above theorem extends a result of Cvetković [14]. A formula for the characteristic polynomial of the generalized composition (Sabidussi's X-join),  $G[H_1, H_2, \dots, H_m]$ , is given in [63] for the case where each  $H_i$  is regular.  $\square$

If  $G \odot H$  is the graph obtained from  $G$  and  $H$  by joining a vertex  $v$  of  $VG$  and a vertex  $w$  of  $VH$  by an edge, then we have the following theorem.

THEOREM 2.3.  $(G \odot H)(\lambda) = G(\lambda)H(\lambda) - G_v(\lambda)H_w(\lambda),$

where  $G_v, H_w$  are the subgraphs of  $G, H$ , respectively, induced by  $VG \setminus \{v\}, VH \setminus \{w\}$ , respectively.

Proof: See [63].  $\square$

A similar result can be achieved by identifying the vertices  $v \in VG$  and  $w \in VH$ . This is called the coalescence of  $G$  and  $H$  and is denoted by  $G \bullet H$ .

THEOREM 2.4.  $(G \bullet H)(\lambda) = G(\lambda)H_w(\lambda) + G_v(\lambda)H(\lambda) - \lambda G_v(\lambda)H_w(\lambda).$

Proof: See [63].  $\square$

We also have the following.

THEOREM 2.5. Suppose that  $VG$  can be partitioned into disjoint sets  $C_1, C_2, \dots, C_m$ , such that the number of vertices in  $C_j$  adjacent to a given vertex in  $C_i$  is independent of the choice of the vertex in  $C_i$ . Let this number be  $c_{ij}$  and let  $C = (c_{ij})$ . Then the characteristic polynomial of  $C$  divides that of  $G$ .

Proof: This is proved in [58], but is a direct consequence of a result in [34].  $\square$

An interesting result relating the characteristic polynomial of a graph, to those of certain of its subgraphs, is given below.

THEOREM 2.6.  $\frac{d}{d\lambda} (G(\lambda)) = \sum_{i=1}^n G_{v_i}(\lambda)$ , where  $VG = \{v_1, v_2, \dots, v_n\}$ .

Proof: See [9]. In this paper Clarke actually uses a different polynomial from the one we are using. However, it is straightforward to obtain the above result from his Corollary 3 to Theorem 5. □

COROLLARY. The vertex-deleted subgraphs of cospectral vertex-transitive graphs are cospectral. Moreover, in the light of the comment in the proof of Theorem 2.1, these vertex-deleted subgraphs also have cospectral complements. □

The following theorem relating the eigenvalues of  $G$  and a subgraph of  $G$  is also worth noting.

THEOREM 2.7. If  $v$  is a vertex of  $G$ , and  $G_v(\lambda) = \Pi(\lambda - \mu_i)$  and  $G(\lambda) = \Pi(\lambda - \lambda_i)$ , then  $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n$ , for a suitable labelling of the eigenvalues.

Proof: This result is a consequence of a well known result in matrix theory, see, for example [4], p. 221. The theorem is noted in [74] and considered further in [47]. □

### 3: CHARACTERISATION OF GRAPHS BY THEIR SPECTRA

In this section we consider the problem of characterising a graph by its spectrum. Although this project for a general graph is doomed to failure by the examples of Section 1 and Section 4, there are a number of types of graphs for which characterisations are known. The results below fall into two categories. One of these says that a graph of a certain type with a particular spectrum belongs to a given set of graphs, while the other says that a graph of a certain type with a particular spectrum can only be a specified graph.

By and large, the results of this section deal with regular graphs. However, the reason for progress being made in these directions is due more to the

fact that the graphs considered have only a small number of distinct eigenvalues, rather than to the regularity, although naturally this latter is of some assistance.

We note that many of the results below have already been surveyed in the papers by Cvetković [16] and Wilson [74], but we include them again here for the sake of completeness. The first few concern line graphs.

THEOREM 3.1. The line graph of the complete graph on  $n$  vertices has as distinct eigenvalues,  $\{-2, n, 2n - 2\}$ . Except for  $n = 8$ , if  $G$  is a regular connected graph on  $n$  vertices with these eigenvalues, then  $G \cong L(K_n)$ .

If  $n = 8$ , then there are three exceptions.

Proof: See [7], [8], [12], [14], [35], [36]. The exceptions are given in [8].  $\square$

THEOREM 3.2. The line graph of the complete bipartite graph  $K_{n,n}$  has as distinct eigenvalues,  $\{-2, n - 2, 2n - 2\}$ . Except for  $n = 4$ , if  $G$  is a regular connected graph on  $2n$  vertices with these eigenvalues, then  $G \cong L(K_{n,n})$ . If  $n = 4$ , then there is one exception.

Proof: See [69].  $\square$

THEOREM 3.3. The line graph of the complete bipartite graph  $K_{m,n}$  has as distinct eigenvalues,  $\{-2, m - 2, n - 2, m + n - 2\}$ .

If  $G$  has  $mn$  vertices and the eigenvalues listed above, then  $G \cong L(K_{m,n})$  if and only if  $G$  has an  $m$ -clique.

Proof: See [21]. A number of other results concerning  $L(K_{m,n})$  and its characterisation are given in this paper.  $\square$

Now a bipartite graph on  $2v$  vertices can be derived from a  $(v, k, \lambda)$  symmetric balanced incomplete block design, where every vertex has degree  $k$  and any two vertices in the same part are adjacent to exactly  $\lambda$  vertices in the other part. Call this graph  $B$ .

THEOREM 3.4. Let  $G$  be a regular connected graph on  $2v$  vertices. The distinct

eigenvalues of  $G$  are given by  $\pm k, \pm\sqrt{k - \lambda}$  if and only if  $G$  is isomorphic to the graph obtained from some symmetric b.i.b.d. with parameters  $(v, k, \lambda)$ .

Proof: See [41]. □

Continuing on the line graph theme, we have

THEOREM 3.5.  $L(B)$  has as distinct eigenvalues  $\{-2, 2k - 2, k - 2 \pm \sqrt{k - \lambda}\}$ .

Except for  $v = 4, k = 3, \lambda = 2$ , if  $G$  is a regular connected graph on  $vk$  vertices with these eigenvalues, then  $G$  is isomorphic to the line graph of a symmetric b.i.b.d. with parameters  $(v, k, \lambda)$ . If  $v = 4, k = 3, \lambda = 2$ , there is one exception.

Proof: See [41]. □

Let  $\pi$  be a projective plane with  $n + 1$  points on a line, i.e.  $\pi$  is of order  $n$ . Let  $H(\pi)$  be the bipartite graph whose vertices are the  $2(n^2 + n + 1)$  points and lines of  $\pi$ , where two vertices are adjacent if and only if one of the vertices is a point, the other is a line, and the point is on the line.

THEOREM 3.6. The line graph of  $H(\pi)$ , for any  $\pi$  of order  $n$ , has as distinct eigenvalues  $\{-2, 2n, n - 1 \pm \sqrt{n}\}$ . If  $G$  is a regular connected graph on  $(n + 1)(n^2 + n + 1)$  vertices with these eigenvalues, then  $G \cong L(H(\pi))$ , for some  $\pi$  of order  $n$ .

Proof: This was proved directly in [37], but is properly a corollary to Theorem 3.5. □

On the way to proving Theorem 3.6, Hoffman also proved the following characterisation.

THEOREM 3.7. A regular connected graph  $G$  on  $2(n^2 + n + 1)$  vertices has as distinct eigenvalues  $\{\pm(n + 1), \pm\sqrt{n}\}$ , if and only if  $G = H(\pi)$  for some projective plane of order  $n$ . □

We now repeat the above process for finite affine planes. If  $\Pi$  is a finite affine plane with  $n$  points on a line, then we define the graph  $H(\Pi)$  whose vertices are the points and lines of  $\Pi$ , with two vertices adjacent if and only if

one is a point, the other a line and the point is on the line.

THEOREM 3.8. For a particular  $\Pi$ ,  $L(H(\Pi))$  has as distinct eigenvalues  $\{-2, 2n - 1, n - 2, \frac{1}{2}[2n - 3 \pm \sqrt{(4n + 1)}]\}$ . If  $G$  is a regular connected graph on  $n^2(n + 1)$  vertices with these eigenvalues, then  $G \cong L(H(\Pi))$ , for some  $\Pi$  of order  $n$ .

Proof: See [42]. □

A graph can also be made from a  $(v, b, r, k, \lambda)$ -balanced incomplete block design. This has  $b + v$  vertices and two vertices are adjacent if and only if one corresponds to a block and the other corresponds to an element in that block. Call this graph  $N$ .

THEOREM 3.9.  $L(N)$  has as distinct eigenvalues  $\{-2, r + k - 2, \frac{1}{2}(r + k - 4) \pm D^{\frac{1}{2}}, k - 2\}$ , where  $D = \frac{1}{4}(r - k)^2 + r - \lambda$ . If  $r + k > 18$ ,  $\lambda = 1$ , and if  $G$  is a regular connected graph on  $vr$  vertices with these eigenvalues, then  $G \cong L(N)$ , for some  $N$ .

Proof: See [24]. □

In a similar vein we have

THEOREM 3.10. The line graph of a Steiner triple system is identified as such by its spectrum if  $r > 15$ .

Proof: This is noted in [20] as a private communication from the author to himself. □

Steiner graphs can be obtained from Steiner triple systems in a natural way by considering the blocks as vertices and saying two vertices are adjacent if the blocks have a common element (see [66]). It can then be shown, [2], that

THEOREM 3.11. For  $s$  sufficiently large, any strong graph on  $\frac{1}{6}s(s - 1)$  vertices, with eigenvalues  $\{-3, \frac{1}{2}(s - 9), \frac{3}{2}(s - 3)\}$  is isomorphic to some Steiner graph of order  $s$ . □

(A strong graph is a graph on  $n$  vertices which is not  $K_n$  or  $\bar{K}_n$ , and whose adjacency matrix  $A = A(G)$  satisfies the following relation

$$[J - 2A - (\rho_1 + 1)I][J - 2A - (\rho_2 + 1)I] = (n - 1 + \rho_1\rho_2)J,$$

where  $J$  is the matrix all of whose entries are 1 and  $\rho_1, \rho_2$  are suitable real numbers with  $\rho_1 \neq \rho_2$ .)

In Theorems 3.1, 3.2, 3.3, 3.5, 3.6, 3.8, and 3.9, which all deal with line graphs, it can be seen that in each case,  $-2$  is an eigenvalue. Further,  $-2$  is the smallest eigenvalue. We now consider generalisations of these observations.

THEOREM 3.12. (a) The minimum eigenvalue of a line graph is greater than, or equal to,  $-2$ .

(b) If  $G$  is connected, the minimum eigenvalue of  $L(G)$  is  $-2$  if and only if either

$$|E(G)| - |V(G)| + 1 > 0 \quad \text{and } G \text{ is bipartite, or}$$

$$|E(G)| - |V(G)| > 0 \quad \text{and } G \text{ is not bipartite.}$$

(c) The minimum eigenvalue of  $L(G)$  is  $-2$  unless every connected component of  $G$  is a tree or has one cycle of odd length and no other cycles.

(d) If the diameter of  $G$  is  $D$ , then the minimum eigenvalue of  $L(G)$  lies between  $-2$  and  $-2 \cos(\pi/(D+1))$ , and these bounds are best possible.

(e) If  $G$  is a regular graph of degree  $r$ , with  $n$  vertices, then

$$L(G)(\lambda) = (\lambda + 2)^{\frac{1}{2}(r-2)n} G(\lambda + 2 - r).$$

(f) Let  $G$  be a bipartite graph with  $n_i$  mutually non-adjacent vertices of degree  $r_i$ ,  $i = 1, 2$ , and  $n_1 \geq n_2$ , then

$$L(G)(\lambda) = (\lambda + 2)^\beta \left\{ \left( -\frac{\alpha_1}{\alpha_2} \right)^{n_1 - n_2} G(\sqrt{\alpha_1\alpha_2}) G(-\sqrt{\alpha_1\alpha_2}) \right\}^{\frac{1}{2}},$$

where  $\alpha_i = \lambda - r_i + 2$ ,  $i = 1, 2$ , and  $\beta = n_1 r_1 - n_1 - n_2$ .

(g) Let  $G$  be a regular connected graph of degree  $\geq 17$  and with smallest eigenvalue  $-2$ , then  $G$  is either a line graph or the complement of the regular graph of degree 1. The number 17 is the best possible.

(h) If  $G = L(H)$  and the minimum degree of  $H$ ,  $d(H)$ , is  $\geq 4$ , then the minimum eigenvalue of  $G$  is  $-2$ . Further, the number of vertices adjacent to both  $u_1$  and  $u_2$ ,  $\Delta(u_1, u_2)$ , is such that for  $u_1, u_2$  non-adjacent,  $\Delta(u_1, u_2) < \deg_G u_i - 2$ ,  $i = 1, 2$ , where  $u_1, u_2 \in VG$ .

(i) If for a graph  $G$ , (a)  $d(G) > 43$ , (b) the minimum eigenvalue is  $-2$ , and (c) for non-adjacent vertices  $u_1, u_2$ , we have  $\Delta(u_1, u_2) < \deg_G u_i - 2$ ,  $i = 1, 2$ , then  $G$  is a line graph.

Proof: The proof of (a) can be found in [39], as is part of (b). The proof of (b), (c), (d) is in [22]. In [59], (e) is proved, and (f) is proved in [16]. No proof of (g) as yet seems to have appeared in print. It is referred to originally in [38] where it is attributed to Hoffman and Ray-Chaudhuri, and then later in [16] and [24] at least. Also in [38], an example due to Seidel is cited (but not given), to show that 17 is best possible. In [54], (h) and (i) are proved. It is expected that the number 43 in (a) is not best possible. □

The following results are along similar lines to the work above, in that they deal with the number  $-2$ .

THEOREM 3.13. If  $T$  is a tree on  $n$  vertices,  $L(\lambda)$  is the characteristic polynomial of the line graph of  $T$ , and  $p$  is a prime, then  $L(-2) \equiv 0 \pmod{p}$  if and only if  $|VT| \equiv 0 \pmod{p}$ .

Proof: See [22]. □

THEOREM 3.14. The only strongly regular graphs with smallest eigenvalue  $-2$ , are the lattice graphs, the triangular graphs, the pseudolattice graphs, the pseudotriangular graphs, the graphs of Petersen, Clebsch and Schläfli, and the complements of the ladder graphs.

Proof: See [65]. The graphs mentioned in this theorem are described in [65] and elsewhere. It should be pointed out that Seidel works with  $(0, -1, 1)$  matrices in this paper, and hence the value 3 in its title. These results can be converted into results for  $(0, 1)$  matrices. Other results on  $(0, -1, 1)$  matrices may be

found in [31], [64], [66], [67]. □

We now see that there are graphs other than line graphs which are characterised by their spectra.

THEOREM 3.15. The graphs on a prime number of vertices, whose automorphism groups are transitive, are identified within this class of graphs by their spectra.

Proof: The eigenvalues of such graphs are given in [72] along with the proof of this result. They are easily obtained since the adjacency matrices of the graphs in question are circulant matrices. It should be noted that, in general, graphs whose adjacency matrices are circulants are not characterised by their spectra. An example of such graphs on 20 vertices is given in the Appendix. □

A cubic lattice graph with characteristic  $n$  ( $n > 1$ ) is a graph whose vertices are all the  $n^3$  ordered triplets on  $n$  symbols, with two triplets adjacent if and only if they differ in exactly one coordinate. These graphs are characterised as follows, where  $\Delta(x, y)$  is the number of vertices adjacent to both  $x$  and  $y$ .

THEOREM 3.16. Except for  $n = 4$ ,  $G$  is the cubic lattice graph with characteristic  $n$ , if and only if its eigenvalues are  $\lambda_f = 3n - 3 - fn$ , with multiplicity  $p_f = \binom{n}{f}(n-1)^f$ ,  $f = 0, 1, 2, 3$  and  $\Delta(x, y) > 1$  for all non-adjacent  $x, y$ .

Proof: See [15] after [48] and [1]. □

A tetrahedral graph is defined to be a graph  $G$ , whose vertices are identified with the  $\binom{n}{3}$  unordered triples on  $n$  symbols, two vertices being adjacent if and only if the corresponding triples have 2 symbols in common.

THEOREM 3.17. If  $G$  is a tetrahedral graph, then (i)  $|VG| = \binom{n}{3}$ , (ii)  $G$  is regular and connected, (iii) the number of vertices at distance 2 from a given vertex  $v$  is  $\frac{3}{2}(n-3)(n-4)$  for all  $v \in VG$ , (iv) the distinct eigenvalues of  $G$  are  $\{-3, 2n-9, n-7, 3n-9\}$ . For  $n > 16$  any graph possessing properties (i)-(iv) is tetrahedral.

Proof: See [6]. □

In [33], Harary and Schwenk pose the problem of determining all graphs whose spectrum consisted entirely of integers. They called these graphs integral graphs.

THEOREM 3.18. The set  $I_r$  of all regular connected integral graphs with a fixed degree  $r$ , is finite.

Proof: See [17]. □

The problem suggested by Theorem 3.18 then is to completely determine the set  $I_r$ . For  $r \leq 2$ , these are  $P_2, C_3, C_4$  and  $C_6$  (see [33]). What if  $r = 3$ ?

THEOREM 3.19. There are thirteen connected cubic integral graphs.

Proof: See [17] and [62]. □

It can also be shown that Cayley graphs of  $\mathbb{Z}_2^n$  always have integral spectra.

At this stage little more seems to be known about integral graphs.

In a similar vein, Doob has tried to determine which graphs have a small number of eigenvalues. Some of this work relates back to earlier theorems concerning line graphs.

THEOREM 3.20. (a)  $G$  has one eigenvalue if and only if  $G = \bar{K}_n$ .

(b)  $G$  has two distinct eigenvalues  $\alpha_1 > \alpha_2$  if and only if each component of  $G$  is  $K_{\alpha_1+1}$  and  $\alpha_2 = -1$ .

(c)  $G$  has eigenvalues  $r, 0, \alpha_2$  if and only if  $G$  is the complement of the union of complete graphs on  $-\alpha_2$  vertices. ( $r$  is the degree of  $G$ .)

(d)  $G$  has eigenvalues  $\pm\alpha, 0$  if and only if  $G = K_{m,n}$  and  $mn = \alpha^2$ .

(e) If  $G$  is regular, then it has eigenvalues  $\pm r, \pm 1$ , if and only if

$G = K_{r+1, r+1}$  minus a 1-factor.

Proof: See [20]. □

THEOREM 3.21. If  $H$  is the graph of a b.i.b.d. and  $G \cong L(H)$ , then

(i)  $G$  has three eigenvalues if and only if the b.i.b.d. is symmetric and

trivial,

(ii)  $G$  has four eigenvalues if and only if the b.i.b.d. is symmetric or trivial, but not both,

(iii)  $G$  has five eigenvalues if and only if the b.i.b.d. is neither symmetric nor trivial.

Proof: See [20]. □

THEOREM 3.22. If  $G$  is a graph with four distinct eigenvalues, the smallest of which is  $-2$ , and  $G \cong L(H)$ , then

(i)  $H$  is strongly regular,

(ii)  $H$  is the graph of a symmetric b.i.b.d.,

or (iii)  $H \cong K_{m,n}$  with  $m > n \geq 2$ .

Proof: See [21]. In fact if  $G$  has four distinct eigenvalues, the smallest of which is  $-2$ , then  $G \cong L(H)$  for some  $H$ , except in a finite number of cases. □

#### 4: COSPECTRAL GRAPHS

In this section we return to a consideration of those graphs which have a cospectral mate. The existence of cospectral graphs was recognised in the paper of Collatz and Sinogowitz [11]. Some of these graphs were rediscovered in [27] and [3] and no doubt elsewhere. In [32] the smallest (in terms of the number of vertices) cospectral graphs and trees were noted. Also in this paper, the smallest cospectral digraphs were listed. We note in passing that more work on cospectral digraphs is done in [46], [53].

In [30], the number of cospectral graphs on 5, 6, 7, 8, 9 vertices are given, while in [67] the eigenvalues of certain strongly regular graphs are listed, for the  $(0, -1, 1)$  adjacency matrix.

In a general sense, it is doubtful whether very much can be said about cospectral graphs. It is possible to find cospectral graphs; cospectral connected graphs; cospectral trees; cospectral forests; cospectral regular graphs; cospectral vertex-transitive graphs; cospectral circulant graphs; cospectral regular graphs -



























