The independence number of graphs with large odd girth

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Abstract. Let G be an r-regular graph of order n and independence number $\alpha(G)$. We show that if G has odd girth $2k+3$ then $\alpha(G) \geq n^{1-1/k} r^{1/k}$. We also prove similar results for graphs which are not regular. Using these results we improve on the lower bound of Monien and Speckenmeyer, for the independence number of a graph of order n and odd girth $2k+3$.

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§1. Introduction

Let G be a triangle–free graph of order n with average degree d, and independence number $\alpha(G)$. There has been great interest in finding good lower bounds for $\alpha(G)$ in terms of d, and producing polynomial–time algorithms which find large independent sets of G. In [1] and [2] Ajtai, Komlós and Szemerédi made a breakthrough in this area when they provided a polynomial algorithm to find an independent set of size at least

$$\alpha(G) \geq \frac{n \log d}{100d} .$$

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A little later this algorithm was sharpened by Griggs in [5] , improving the constant from 100^{-1} to 2.4^{-1}. Shearer, in [8] , improved this bound still further to give that

$$\alpha(G) \geq n\left[\frac{d(\log d) - d + 1}{(d-1)^2}\right] .$$

In [8] besides extending this result to take the degree sequence of the graph into account Shearer also considered what could be said for graphs of larger odd girth. He proved the following theorem.

Theorem A. Let G be a graph of order n with degree sequence $d_1, d_2, \ldots, d_n$. Suppose that G contains no 3 or 5 cycles. Let n_{11} be the number of pairs of adjacent vertices of degree 1 in G. Let $f(0) = 1$, $f(1) = 4/7$ and $f(d) = [1 + (d^2 - d)f(d-1)](d^2+1)^{-1}$ when $d \geq 2$. Then

$$\alpha(G) \geq \sum_{i=1}^{n} f(d_i) - n_{11}/7 .$$

□

The results of this paper are designed to deal with the case when the average degree of the graph is large. We shall prove that an r-regular graph without 3 and 5 cycles has an independence number of at least

$$\alpha(G) \geq \sqrt{\frac{nr}{6}} .$$

Indeed we shall provide a polynomial–time algorithm to produce such a set of independent vertices. More generally,we shall show that an r-regular graph with odd girth $2k+3$ has an independent set of size at least

$$\alpha(G) \geq c_k n^{1-1/k}\, r^{1/k} .$$

This technique can also be used to give new bounds for the independence number of general graphs of a given odd girth. We shall prove some similar bounds to those we prove for regular graphs in terms of a measure of the concentration of edges.

Monien and Speckenmeyer in [6] investigated the special Ramsey number $r_k(q)$, the largest number of vertices in a graph with odd girth at least $2k+3$, but not containing an independent set of size $q+1$. They showed that

$$r_k(q) \leq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k}{k+2} q .$$

Combining our new bound with that of Shearer we show a new bound for the Ramsey number $r_k(q)$

$$r_k(q) \leq \left(\frac{k}{\ln q}\right)^{\frac{1}{k}} q^{\frac{k+1}{k}}$$

improving the previous bound provided q is large.

§2. The independence number of regular graphs

In this section we shall introduce the basic algorithmic method we shall use find large independent sets in graphs with large odd girth at. To illustrate the ideas behind this algorithm we shall first prove our results for graphs of odd girth at least 7. Dealing with graphs with larger odd girth will simply require a generalisation of this argument.

Theorem 1 Let G be a graph of order n containing containing no 3 or 5 cycles with average degree $\bar{\mathrm{d}}(G)$ and minimal degree $\delta \geq 2\bar{\mathrm{d}}(G)/3$. Then

$$\alpha(G) \geq \frac{1}{\sqrt{2}}\sqrt{n\left(\delta - \frac{2\bar{\mathrm{d}}(G)}{3}\right)}$$

and there is a polynomial–time algorithm that finds an independent set of at least this size.

Proof. Let

$$m = \frac{1}{2\sqrt{2}}\sqrt{n\left(\delta - \frac{2}{3}\bar{\mathrm{d}}(G)\right)^{-1}}$$

We begin by trying to greedy–colour the vertices of G with m colours. In other words we take the vertices one at a time and for each vertex use the smallest available colour. If no colour is available we ignore that vertex and proceed to the next. Firstly suppose that this greedy colouring colours at least $n/4$ vertices. Then, clearly, one of the colour classes will have size at least

$$\frac{n}{4m} = \frac{1}{\sqrt{2}}\sqrt{n\left(\delta - \frac{2\bar{\mathrm{d}}(G)}{3}\right)}$$

and we have an independent set to satisfy the theorem.

Suppose then that we are not so successful and that $g' \geq 3n/4$ vertices remain uncoloured. Let $A_1, A_2, \ldots, A_m$ be the greedy colour classes. Consider the following algorithm **SHUFFLE**(c) for a real parameter c.

Algorithm: SHUFFLE(c)

- $V(G') = V(G) \setminus \bigcup_{i=1}^{m} A_i$;
- Choose $v \in V(G')$;
- Let $I = \Gamma_{G'}(v)$;
- Let $N_i(v) = \Gamma_G(I) \cap A_i$ for $i = 1, \ldots, m$;
- If there is an i for which $|I| > |N_i(v)|$ then $A_i = A_i \setminus N_i(v) \cup I$;
- Repeat until $|\bigcup_{i=1}^{m} A_i| \geq c$ or until every vertex of G' has been chosen since the last time G' changed.

As usual set $\Gamma_G(I) = \{v : vi \in E(G) \text{ for some } i \in I\}$. Notice that since I is the neighbourhood of a vertex and G is triangle free, I is an independent set. Thus each A_i remains an independent set throughout the algorithm. We apply **SHUFFLE**($n/4$) to the graph.

Consider the situation when the algorithm stops. Either the greedy–colour classes $A_1, A_2, \ldots, A_m$ comprise at least $n/4$ vertices and we may argue as before, or for any uncoloured vertex $v \in V(G')$

$$|N_i(v)| \geq |\Gamma_{G'}(v)| \qquad \text{for } 1 \leq i \leq m \ .$$

Suppose the latter holds. Then, given a vertex $v \in V(G')$, certainly each $N_i(v)$ is an independent set, since each is a subset of a colour class, but in fact $\bigcup_{i=1}^{m} N_i(v)$ is an independent set. To prove this we need only show that there can be no edges between a vertex of $N_i(v)$ and $N_j(v)$ for $i \neq j$.

Suppose that we have such an edge ab for $a \in N_i(v)$ and $b \in N_j(v)$. Let $I = \Gamma_{G'}(v)$ and consider $\Gamma_G(a) \cap \Gamma_G(b) \cap I$.

Firstly suppose that $\Gamma_G(a) \cap \Gamma_G(b) \cap I \neq \emptyset$, containing a vertex, c say. Then vertices a, b, c form a triangle in G, contradicting the odd girth of G. Otherwise, since by construction $I_G(a) \cap I$ and $I_G(b) \cap I$ are non–empty, there exist distinct vertices $c \in I_G(a) \cap I$ and $d \in I_G(b) \cap I$. Then a, c, v, d, b form a 5-cycle in G, giving the required contradiction.

Now, if we choose a vertex v of maximal degree in G', we certainly have $|\Gamma_{G'}(v)| \geq \bar{\mathrm{d}}(G')$, and since $|N_i(v)| \geq |\Gamma_{G'}(v)| \geq \bar{\mathrm{d}}(G')$ we have that

$$\left| \bigcup_{i=1}^{m} N_i(v) \right| \geq m\bar{\mathrm{d}}(G') \ .$$

Hence the algorithm is guaranteed to find an independent set of size at least

$$\min\left\{\frac{1}{\sqrt{2}}\sqrt{n(\delta-\frac{2\bar{\mathrm{d}}(G)}{3})}, m\bar{\mathrm{d}}(G')\right\} .$$

It remains only to show that $\bar{\mathrm{d}}(G')$ cannot be too small. We do this with a simple counting argument. Let $H' = G\backslash G'$ and let us count the number of edges in G $e(G)$. Then we see that

$$e(G) = \frac{n\bar{\mathrm{d}}(G)}{2} \geq \frac{(n-g')\bar{\mathrm{d}}(H')}{2} + g'\delta - \frac{g'\bar{\mathrm{d}}(G')}{2} .$$

Thus, rearranging this inequality we have

$$\begin{aligned}\bar{\mathrm{d}}(G') &\geq 2\delta + \frac{(n-g')}{g'}\bar{\mathrm{d}}(H') - \frac{n}{g'}\bar{\mathrm{d}}(G)\\ &\geq 2\delta - \frac{n}{g'}\bar{\mathrm{d}}(G) .\end{aligned}$$

The right hand side of this inequality is increasing with g'. Hence, since $g' \geq 3n/4$,

$$\bar{\mathrm{d}}(G') \geq 2\delta - \frac{4\bar{\mathrm{d}}(G)}{3}$$

and so using this bound we see that

$$m\bar{\mathrm{d}}(G) \geq \frac{1}{\sqrt{2}}\sqrt{n\left(\delta - \frac{2\bar{\mathrm{d}}(G)}{3}\right)} .$$

□

Thus Theorem 1 shows that provided the minimal degree is not too small, there is a large independent set in the graph. In particular, we may apply this result to the case when G is a regular graph.

Theorem 2 Let G be an r–regular graph of order n with no 3 or 5 cycles. Then

$$\alpha(G) \geq \sqrt{\frac{nr}{6}} .$$

□

Using a similar technique, but applying the algorithm **SHUFFLE** recursively we can extend Theorem 1 to deal with graphs known to have larger odd girth.

Theorem 3 Let G be a graph of order n with odd girth $2k+3$ ($k \geq 2$) and minimal degree $\delta(G) \geq \frac{2\bar{\mathrm{d}}(G)}{3}$. Then

$$\alpha(G) \geq \left(\frac{n}{4(k-1)}\right)^{\frac{k-1}{k}} \left(2\delta - \frac{4\bar{\mathrm{d}}(G)}{3}\right)^{\frac{1}{k}}.$$

Proof. To construct our independent set we mimic the proof of Theorem 1 , but this time we choose

$$m = \left(\frac{n}{8(k-1)}\left(\delta - \frac{2\bar{\mathrm{d}}(G)}{3}\right)^{-1}\right)^{\frac{1}{k}}.$$

Firstly let us greedily colour the vertices of G just as we did in Theorem 1 but this time with $(k-1)m$ colours. Clearly if any sm colour classes together contain at least $sn/4(k-1)$ vertices for some $1 \leq s \leq (k-1)$ then immediately we have a colour class of size at least

$$\frac{n}{4(k-1)m} = \left(\frac{n}{4(k-1)}\right)^{\frac{k-1}{k}} \left(2\delta - \frac{4\bar{\mathrm{d}}(G)}{3}\right)^{\frac{1}{k}}$$

as required. If not then, as before let, $A_1, A_2, \ldots, A_{(k-1)m}$ be the greedy–colour classes.

For $x, y \in V(G)$ let $d_G(x,y)$ be the usual graph–distance, the minimum number of edges in a path joining x to y in G.

For each integer $1 \leq s \leq (k-1)$ and real c we define a new algorithm similar to **SHUFFLE**.

Algorithm: SHUFFLE(c, s)

- Let $V(G'_s) = V(G) \backslash \bigcup_{i=1}^{sm} A_i$
- Choose $v \in V(G'_s)$
- Let $I(v) = \{u; d_{G'_s}(u,v) = k-s\}$
- Let $N_i^s(v) = \Gamma_G(I(v)) \cap A_i \qquad$ for $(s-1)m+1 \leq i \leq sm$
- If there is some $(s-1)m+1 \leq j \leq sm$ for which $|I(v)| > |N_j^s(v)|$ then let $A_i = A_i \backslash N_j^s(v) \cup \ I(v)$
- Repeat from the beginning until $|\bigcup_{i=1}^{sm} A_i| \geq c$ or until each vertex of G'_s has been chosen since the last time G'_s changed.

Now let us show that, just as our neighbourhoods in **SHUFFLE** form a large independent set, here $\bigcup_{i=(s-1)m+1}^{sm} N_i^s(v)$ is an independent set. To do this, as before we show that there can be no edge between a vertex a of N_i^s and a vertex b of N_j^s, $i \neq j$. Clearly $d_G(v,a) = d_G(v,b) = k - s + 1$. Thus consider paths of minimal length joining a and b to v, and let p be the vertex furthest from v at which these paths intersect (certainly since they each pass through v there is some intersection). By the minimality of the paths we must have $1 \leq d_G(a,p) = d_G(b,p) \leq k - s + 1$. Thus if $ab \in E(G)$ $a \ldots p \ldots b$ forms a cycle of length $3 \leq 2d_G(a,p) + 1 \leq 2k + 1$ contradicting the odd girth of G.

Indeed similarly to the original **SHUFFLE** algorithm, on completion the new algorithm **SHUFFLE**(c,s) either produces a greedy–colouring of at least c vertices with sm colours, or ensures that for any vertex $v \in G'_s$ $|N_i^s| \geq |I(v)|$ for each $(s-1)m + 1 \leq i \leq sm$.

Let us now define the following algorithm which uses **SHUFFLE**(c,s):

Algorithm: SHUFFLE*(c)

- do $i = k - 1$ to 1
- do $j = i$ to $k - 1$
- **SHUFFLE**(jc, j)
- continue

Let us apply **SHUFFLE***$(n/(4k-4))$ to G. Then on completion either some collection of sm colour classes will contain at least $sn/(4k-4)$ vertices (for some $1 \leq s \leq (k-1)$) and we immediately have a large independent set, or at least $n - \frac{(k-1)n}{4(k-1)} = \frac{3n}{4}$ vertices remain uncoloured, thus $\left|V(G'_{(k-1)})\right| \geq 3n/4$, and for any uncoloured vertex $v \in G'_{(k-1)}$

$$\sum_{i=(s-1)m+1}^{sm} |N_i^s(v)| \geq m^{(k-s)} |\Gamma_{G'_{k-1}}(v)| \qquad 1 \leq s \leq (k-1) \; .$$

Now if we choose v to be a vertex of $V(G'_{k-1})$ with degree at least $\bar{\mathrm{d}}(G'_{(k-1)})$ then $\bigcup_{i=1}^{m} N_i^1(v)$ is an independent set of size

$$\sum_{i=1}^{m} |N_i^1(v)| \geq m^{(k-1)} \bar{\mathrm{d}}(G'_{(k-1)}) \; .$$

Thus the algorithm guarantees to find an independent set of size

$$\min\left\{\frac{n}{4(k-1)m}, m^{k-1}\bar{\mathrm{d}}(G'_{(k-1)})\right\}.$$

It remains only to reapply the argument used in the proof of Theorem 1 to show that

$$\bar{\mathrm{d}}(G'_{(k-1)}) \geq 2\delta(G) - \frac{4\bar{\mathrm{d}}(G)}{3}$$

and hence that we have an independent set of size

$$\left(\frac{n}{4(k-1)}\right)^{\frac{k-1}{k}}\left(2\delta - \frac{4\bar{\mathrm{d}}(G)}{3}\right)^{\frac{1}{k}}.$$

□

Applying this bound when the graph is r-regular graph, we immediately have an analogous result to Theorem 2 .

Theorem 4 Let G be an r-regular graph of order n and odd girth $2k+3$ ($k \geq 2$). Then

$$\alpha(G) \geq c_k\, r^{1/k}\, n^{1-1/k}$$

where

$$c_k = \left(\frac{2}{3}\right)^{1/k} (4(k-1))^{-(k-1)/k}.$$

□

§3. Further independence results

The use of the method of Section 2 is not solely limited to graphs which are almost regular. In the non–regular case we can still find bounds for the independence number in terms of the odd girth, but instead of the average degree of the graph we have to use another measure of the concentration of edges.

Theorem 5 Let G be a graph of order n with odd girth at least $2k+3$ ($k \geq 2$), and let

$$\Delta_0 = \min\{\Delta(H) : H \subset G, |V(H)| \geq n/k\}$$

and

$$\bar{d}_0 = \min\{\bar{\mathrm{d}}(H) : H \subset G, |V(H)| \geq n/k\} .$$

Then

$$\alpha(G) \geq \max\left\{ \frac{n \log \bar{d}_0}{2.4k\bar{d}_0}, \left(\frac{n}{k}\right)^{1-1/k} \Delta_0^{1/k} \right\} .$$

Proof. Firstly as before we can produce an independent set of size at least

$$\frac{n \log \bar{d}_0}{2.4k\bar{d}_0}$$

by applying the Griggs' algorithm to a subgraph H which achieves $\bar{d}_0$ as its average degree.

To produce an independent set of the other size we mimic the proof of Theorem 3 but this time we choose

$$m = \left(\frac{n}{k\Delta_0}\right)^{\frac{1}{k}} .$$

Let us colour the vertices of G with $(k-1)m$ colours. Clearly if any sm colour classes together contain at least sn/k vertices ($1 \leq s \leq (k-1)$) then immediately we have a colour class of size at least

$$\frac{n}{km} = \left(\frac{n}{k}\right)^{1-\frac{1}{k}} \Delta_0^{\frac{1}{k}} .$$

If not, then as before let $A_1, A_2, \ldots, A_{(k-1)m}$ be the greedy–colour classes. Let us now apply **SHUFFLE***(n/k).

When the algorithm stops, either one of the colour classes provides us with the large independent set we desire or for any uncoloured vertex $v \in G'_{(k-1)}$ we have

$$\sum_{i=(s-1)m+1}^{sm} |N_i^s(v)| \geq m^{(k-s)} |\Gamma_{G'_{k-1}}(v)| \qquad 1 \leq s \leq (k-1) .$$

Now since, $|V(G'_{(k-1)})| \geq n/k$, by definition of Δ_0 we must be able to choose a v so that $|\Gamma_{G'_{(k-1)}}(v)| \geq \Delta_0$ and for this choice we have that $\bigcup_{i=1}^{m} N_i^1(v)$ is an independent set of size

$$\sum_{i=1}^{m} |N_i^1(v)| \geq m^{(k-1)} \Delta_0 = \left(\frac{n}{k}\right)^{1-\frac{1}{k}} \Delta_0^{\frac{1}{k}} .$$

□

In particular, when $k = 2$ and the graph has no 3 or 5 cycles we have an analogous result to Theorem 2 and an extension of Shearer's result, Theorem A.

Theorem 6 Let G be a graph of order n having no 3 or 5 cycle, and let

$$\Delta_0 = \min\left\{\Delta(H) : H \subset G, |V(H)| \geq n/2\right\}$$

and

$$\bar{d}_0 = \min\left\{\bar{\mathrm{d}}(H) : H \subset G, |V(H)| \geq n/2\right\} .$$

Then

$$\alpha(G) \geq \max\left\{\frac{n \log \bar{d}_0}{4.8\bar{d}_0}, \sqrt{\frac{n\Delta_0}{2}}\right\} .$$

These results lead directly to a general lower bound for the the independence number of a graph in terms of its order and odd girth simply by minimising the bounds in Theorem 5.

Corollary 7 Let G be a graph of order n with odd girth at least $2k+3$ ($k \geq 2$). Then

$$\alpha(G) \geq \left(\frac{n}{k}\right)^{\frac{k}{k+1}} (\log n)^{\frac{1}{k+1}} .$$

□

Looking at the problem the other way round, Monien and Speckenmeyer in [6] proved a bound for the Ramsey number $r_k(q)$, the largest number of vertices in a graph with odd girth $2k + 3$ and independence number at most q. Monien and Speckenmeyer showed that

$$r_k(q) \leq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k}{k+2} q .$$

Using Theorem 5 once again, we can improve their upper bound.

Theorem 8 Let $k \geq 2$. Then

$$r_k(q) \leq \left(\frac{k^{k+2}}{(k+1)\log q} \right)^{1/k} q^{\frac{k+1}{k}} .$$

□

Concluding remarks

A *vertex cover* of a graph G is a set of vertices U so that for every edge $ab \in E(G)$ a or b is a member of U. We shall write $\lambda(G)$ for the minimum size of a vertex cover of G.

The *vertex cover problem* then is to find a vertex cover U of G in polynomial time, so that $|U|/\lambda(G)$ is as small as possible. The main result of Monien and Spekenmeyer's paper [6] is to produce an algorithm to find a vertex cover so that this ratio is always at most $2 - \log\log n/\log n$. The bound on the effectiveness of the algorithm depends entirely on the bound which they generate for $r_q(k)$. It is thus unfortunate that although our bound improves on their bound it does so only when q is too large to improve the bound on the algorithms effectiveness.

However it should be noted that in generating the bound for Theorem 8 for one of the bounds we assumed only that the graph was triangle free. Clearly if some result similar to those contained in this chapter could give a better bound on the independence number of a graph with large odd girth and small average degree an improvement on the bound for $r_q(k)$ and perhaps the effectiveness of Monien and Speckenmeyer's algorithm would be immediate.

This improvement could also be passed on to various other polynomial–time algorithms which use the vertex cover algorithm including, for instance, the algorithm of Blum (see [3] and [4]) to colour a 3–chromatic graph in polynomial–time in at most $O(n^{3/8})$ colours. We hope in the future to extend our results to the small degree case.

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