

ALGEBRAIC MATCHING THEORY

C. D. Godsil ¹

Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1
chris@bilby.uwaterloo.ca

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Abstract: The number of vertices missed by a maximum matching in a graph G is the multiplicity of zero as a root of the matchings polynomial $\mu(G, x)$ of G , and hence many results in matching theory can be expressed in terms of this multiplicity. Thus, if $\text{mult}(\theta, G)$ denotes the multiplicity of θ as a zero of $\mu(G, x)$, then Gallai's lemma is equivalent to the assertion that if $\text{mult}(\theta, G \setminus u) < \text{mult}(\theta, G)$ for each vertex u of G , then $\text{mult}(\theta, G) = 1$.

This paper extends a number of results in matching theory to results concerning $\text{mult}(\theta, G)$, where θ is not necessarily zero. If P is a path in G then $G \setminus P$ denotes the graph got by deleting the vertices of P from G . We prove that $\text{mult}(\theta, G \setminus P) \geq \text{mult}(\theta, G) - 1$, and we say P is θ -essential when equality holds. We show that if, all paths in G are θ -essential, then $\text{mult}(\theta, G) = 1$. We define G to be θ -critical if all vertices in G are θ -essential and $\text{mult}(\theta, G) = 1$. We prove that if $\text{mult}(\theta, G) = k$ then there is an induced subgraph H with exactly k θ -critical components, and the vertices in $G \setminus H$ are covered by k disjoint paths.

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1. Introduction

A k -matching in a graph G is a matching with exactly k edges and the number of k -matchings in G is denoted by $p(G, k)$. If $n = |V(G)|$ we define the matchings polynomial $\mu(G, x)$ by

$$\mu(G, x) := \sum_{k \geq 0} (-1)^k p(G, k) x^{n-2k}.$$

(Here $p(G, 0) = 1$.) By way of example, the matchings polynomial of the path on four vertices is $x^4 - 3x^2 + 1$. The matchings polynomial is related to the characteristic polynomial $\phi(G, x)$ of G , which is defined to be the characteristic polynomial of the adjacency matrix of G . In particular $\phi(G, x) = \mu(G, x)$ if and only if G is a forest [4: Corollary 4.2]. Also the matchings polynomial of any connected graph is a factor of the characteristic polynomial of some tree. (For this, see Theorem 2.2 below.)

Let $\text{mult}(\theta, G)$ denote the multiplicity of θ as a zero of $\mu(G, x)$. If $\theta = 0$ then $\text{mult}(\theta, G)$ is the number of vertices in G missed by a maximum matching. Consequently many classical results in the theory of matchings provide information related to $\text{mult}(0, G)$. We refer in particular to Gallai's lemma and the Edmonds-Gallai structure theorem, which we now discuss briefly.

A vertex u of G is θ -essential if $\text{mult}(\theta, G \setminus u) < \text{mult}(\theta, G)$. So a vertex is 0-essential if and only if it is missed by some maximum matching of G . Gallai's lemma is the assertion that if G is connected, $\theta = 0$ and every vertex is θ -essential then $\text{mult}(\theta, G) = 1$. (A more traditional expression of this result is given in [8: §3.1].) A vertex is θ -special if it is not θ -essential but has a neighbour which is θ -essential. The Edmonds-Gallai structure in large part reduces to the assertion that if $\theta = 0$ and v is a θ -special vertex in G then a vertex u is θ -essential in G and if and only if it is θ -essential in $G \setminus v$. (For more information, see [8: §3.2].) One aim of the present paper is to investigate the extent to which these results are true when $\theta \neq 0$.

There is a second source of motivation for our work. Heilman and Lieb proved that if G has a Hamilton path then all zeros of $\mu(G, x)$ are simple. (This is an easy consequence of Corollary 2.5 below.) Since all known vertex-transitive graphs have Hamilton paths we are lead to ask whether there is a vertex-transitive graph G such that $\mu(G, x)$ has a multiple zero. As we will see, it is easy to show that if θ is a zero of $\mu(G, x)$ and G is vertex-transitive then every vertex of G is θ -essential. Hence, if we could prove Gallai's lemma for general zeros of the matchings polynomial, we would have a negative answer to this question.

2. Identities

The first result provides the basic properties of the matchings polynomial $\mu(G, x)$. We write $u \sim v$ to denote that the vertex u is adjacent to the vertex v . For the details see, for example, [6: Theorem 1.1].

2.1 Theorem. *The matchings polynomial satisfies the following identities:*

- (a) $\mu(G \cup H, x) = \mu(G, x) \mu(H, x)$,
- (b) $\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus uv, x)$ if $e = \{u, v\}$ is an edge of G ,
- (c) $\mu(G, x) = x \mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x)$, if $u \in V(G)$,
- (d) $\frac{d}{dx} \mu(G, x) = \sum_{i \in V(G)} \mu(G \setminus i, x)$. □

Let G be a graph with a vertex u . By $\mathcal{P}(u)$ we denote the set of paths in G which start at u . The *path tree* $T(G, u)$ of G relative to u has $\mathcal{P}(u)$ as its vertex set, and two paths are adjacent if one is a maximal proper subpath of the other. Note that each path in $\mathcal{P}(u)$ determines a path starting with u in $T(G, u)$ and with same length. We will usually denote them by the same symbol. The following result is taken from [6: Theorem 6.1.1].

2.2 Theorem. *Let u be a vertex in the graph G and let $T = T(G, u)$ be the path tree of G with respect to u . Then*

$$\frac{\mu(G \setminus u, x)}{\mu(G, x)} = \frac{\mu(T \setminus u, x)}{\mu(T, x)}$$

and, if G is connected, then $\mu(G, x)$ divides $\mu(T, x)$. □

Because the matchings polynomial of a tree is equal to the characteristic polynomial of its adjacency matrix, its zeros are real; consequently Theorem 2.2 implies that the zeros of the matchings polynomial of G are real, and also that they are interlaced by the zeros of $\mu(G \setminus u, x)$, for any vertex u . (By interlace, we mean that, between any two zeros of $\mu(G, x)$, there is a zero of $\mu(G \setminus u, x)$. This implies in particular that the multiplicity of a zero θ in $\mu(G, x)$ and $\mu(G \setminus u, x)$ can differ by at most one.) For a more extensive discussion of these matters, see [6: §6.1].

We will need a strengthening of the first claim in Theorem 2.2.

2.3 Corollary. *Let u be a vertex in the graph G and let $T = T(G, u)$ be the path tree of G with respect to u . If $P \in \mathcal{P}(u)$ then*

$$\frac{\mu(G \setminus P, x)}{\mu(G, x)} = \frac{\mu(T \setminus P, x)}{\mu(T, x)}.$$

Proof. We proceed by induction on the number of vertices in P . If P has only one vertex, we appeal to the theorem. Suppose then that P has at least two vertices in it, and that v is the end vertex of P other than u . Let Q be the path $P \setminus v$ and let H denote $G \setminus Q$. Then

$$\frac{\mu(G \setminus P, x)}{\mu(G, x)} = \frac{\mu(G \setminus P, x)}{\mu(G \setminus Q, x)} \frac{\mu(G \setminus Q, x)}{\mu(G, x)} = \frac{\mu(T(H, v) \setminus v, x)}{\mu(T(H, v), x)} \frac{\mu(T \setminus Q, x)}{\mu(T, x)},$$

where the second equality follows by induction. Now $T(H, v)$ is one component of $T(G, u) \setminus Q$, and if we delete the vertex v from this component from $T(G, u) \setminus Q$, the graph that results is $T(G, u) \setminus P$. Consequently

$$\frac{\mu(T(H, v) \setminus v, x)}{\mu(T(H, v), x)} = \frac{\mu(T \setminus P, x)}{\mu(T \setminus Q, x)}.$$

The results follows immediately from this. □

Let $\mathcal{P}(u, v)$ denote the set of paths in G which start at u and finish at v . The following result will be one of our main tools. It is a special case of [7: Theorem 6.3].

2.4 Lemma (Heilmann and Lieb). *Let u and v be vertices in the graph G . Then*

$$\mu(G \setminus u, x) \mu(G \setminus v, x) - \mu(G, x) \mu(G \setminus uv, x) = \sum_{P \in \mathcal{P}(u, v)} \mu(G \setminus P, x)^2. \quad \square$$

This lemma has a number of important consequences. In [5: Section 4] it is used to show that $\text{mult}(\theta, G)$ is a lower bound on the number of paths needed to cover the vertices of G , and that the number of distinct zeros of $\mu(G, x)$ is an upper bound on the length of a longest path. For our immediate purposes, the following will be the most useful.

2.5 Corollary. *If P is a path in the graph G then $\mu(G \setminus P, x) / \mu(G, x)$ has only simple poles. In other words, for any zero θ of $\mu(G, x)$ we have*

$$\text{mult}(\theta, G \setminus P) \geq \text{mult}(\theta, G) - 1.$$

Proof. Suppose $k = \text{mult}(\theta, G)$. Then, by interlacing, $\text{mult}(\theta, G \setminus u) \geq k - 1$ for any vertex u of G and $\text{mult}(\theta, G \setminus uv) \geq k - 2$. Hence the multiplicity of θ as a zero of

$$\mu(G \setminus u, x) \mu(G \setminus v, x) - \mu(G, x) \mu(G \setminus uv, x)$$

is at least $2k - 2$. It follows from Lemma 2.4 that $\text{mult}(\theta, G \setminus P) \geq k - 1$ for any path P in $\mathcal{P}(u, v)$. □

3. Essential Vertices and Paths

Let θ be a zero of $\mu(G, x)$. A path P of G is θ -essential if $\text{mult}(\theta, G \setminus P) < \text{mult}(\theta, G)$. (We will often be concerned with the case where P is a single vertex.) A vertex is θ -special if it is not θ -essential and is adjacent to an θ -essential vertex. A graph is θ -primitive if and only if every vertex is θ -essential and it is θ -critical if it is θ -primitive and $\text{mult}(\theta, G) = 1$. (When θ is determined by the context we will often drop the prefix ‘ θ -’ from these expressions.) If $\theta = 0$ then a θ -critical graph is the same thing as a factor-critical graph.

The next result implies that a vertex-transitive graph is θ -primitive for any zero θ of its matchings polynomial.

3.1 Lemma. *Any graph has at least one essential vertex.*

Proof. Let θ be a zero of $\mu(G, x)$ with multiplicity k . Then θ has multiplicity $k - 1$ as a zero of $\mu'(G, x)$. Since

$$\mu'(G, x) = \sum_{u \in V(G)} \mu(G \setminus u, x)$$

we see that if $\text{mult}(\theta, G \setminus u) \geq k$ for all vertices u of G then θ must have multiplicity at least k as a zero of $\mu'(G, x)$. \square

3.2 Lemma. *If $\theta \neq 0$ then any θ -essential vertex u has a neighbour v such that the path uv is essential.*

Proof. Assume $\theta \neq 0$ and let u be a θ -essential vertex. Since

$$\mu(G, x) = x \mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x)$$

we see that if $\text{mult}(\theta, G \setminus ui) \geq \text{mult}(\theta, G)$ for all neighbours i of u then $\text{mult}(\theta, G \setminus u) \geq \text{mult}(\theta, G)$. \square

Note that the vertex v is not essential in $G \setminus u$. However it follows from the next lemma that the vertex v in the above lemma must be essential in G ; accordingly if $\theta \neq 0$ then any essential vertex must have an essential neighbour.

3.3 Lemma. *If v is not an essential vertex of G then no path with v as an end-vertex is essential.*

Proof. Assume $k = \text{mult}(\theta, G)$. If v is not essential then $\text{mult}(\theta, G \setminus v) \geq k$ and so, for any vertex u not equal to v , the multiplicity of θ as a zero of

$$\mu(G \setminus u, x) \mu(G \setminus v, x) - \mu(G, x) \mu(G \setminus uv, x)$$

is at least $2k - 1$. By Lemma 2.4 we deduce that it is at least $2k$ and that $\text{mult}(\theta, G \setminus P) \geq k$ for all paths P in $\mathcal{P}(v)$. \square

We now need some more notation. Suppose that G is a graph and θ is a zero of $\mu(G, x)$ with positive multiplicity k . A vertex u of G is θ -positive if $\text{mult}(\theta, G \setminus u) = k + 1$ and θ -neutral if $\text{mult}(\theta, G \setminus u) = k$. (The ‘negative’ vertices will still be referred to as essential.) Note that, by interlacing, $\text{mult}(\theta, G \setminus u)$ cannot be greater than $k + 1$.

3.4 Lemma. *Let G be a graph and u a vertex in G which is not essential. Then u is positive in G if and only if some neighbour of it is essential in $G \setminus u$.*

Proof. From Theorem 2.1(c) we have

$$\mu(G, x) = x \mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x). \quad (3.1)$$

If $\text{mult}(\theta, G \setminus u) = k + 1$ and $\text{mult}(\theta, G \setminus ui) \geq k + 1$ for all neighbours i of u then it follows that $\text{mult}(\theta, G) \geq k + 1$ and u is not positive.

On the other hand, suppose u is not essential in G and v is a neighbour of u which is essential in $G \setminus u$. From the previous lemma we see that the path uv is not essential and thus $\text{mult}(\theta, G \setminus uv) \geq \text{mult}(\theta, G)$. As v is essential in $G \setminus u$ it follows that $\text{mult}(\theta, G \setminus u) > \text{mult}(\theta, G)$. \square

We say that S is an *extremal* subtree of the tree T if S is a component of $T \setminus v$ for some vertex v of G .

3.5 Lemma. *Let S be an extremal subtree of T that is inclusion-minimal subject to the condition that $\text{mult}(\theta, S) \neq 0$, and let v be the vertex of T such that S is a component of $T \setminus v$. Then v is θ -positive in T .*

Proof. Let u be the vertex of S adjacent to v and let e be the edge $\{u, v\}$. Then $T \setminus e$ has exactly two components, one of which is S . Denote the other by R .

By hypothesis $\text{mult}(\theta, S') = 0$ for any component S' of $S \setminus u$, therefore $\text{mult}(\theta, S \setminus u) = 0$ by Theorem 2.1(a) and so u is essential in S . Since S is a component of $T \setminus v$ it follows that u is essential in $T \setminus v$. If we can show that v is not essential then v must be positive in T , by the previous lemma.

Suppose $\text{mult}(\theta, T) = m$. By interlacing $\text{mult}(\theta, T \setminus u) \geq m - 1$ and, as

$$\text{mult}(\theta, T \setminus u) = \text{mult}(\theta, R) + \text{mult}(\theta, S \setminus u) = \text{mult}(\theta, R),$$

we find that $\text{mult}(\theta, R) \geq m - 1$. By parts (a) and (b) of Theorem 2.1 we have

$$\mu(T, x) = \mu(R, x) \mu(S, x) - \mu(R \setminus v, x) \mu(S \setminus u, x)$$

and so, since the multiplicity of θ as a zero of $\mu(R, x) \mu(S, x)$ is at least m , we deduce that the multiplicity of θ as a zero of $\mu(R \setminus v, x) \mu(S \setminus u, x)$ is at least m . Since $\text{mult}(\theta, S \setminus u) = 0$, it follows that $\text{mult}(\theta, R \setminus v) \geq m$. On the other hand

$$\text{mult}(\theta, T \setminus v) = \text{mult}(\theta, R \setminus v) + \text{mult}(\theta, S) = \text{mult}(\theta, R \setminus v) + 1,$$

therefore $\text{mult}(\theta, T \setminus v) \geq m + 1$ and v is positive in T . □

3.6 Corollary (Neumaier). *Let T be a tree and let θ be a zero of $\mu(T, x)$. The following assertions are equivalent:*

- (a) $\text{mult}(\theta, S) = 0$ for all extremal subtrees of T ,
- (b) T is θ -critical,
- (c) T is θ -primitive.

Proof. Since $T \setminus v$ is a disjoint union of extremal subtrees for any vertex v in T , we see that if (a) holds then $\text{mult}(\theta, T \setminus v) = 0$ for any vertex v . Hence T is θ -critical and therefore it is also θ -primitive. If T is θ -primitive then no vertex in T is θ -positive, whence Lemma 3.5 implies that (a) holds. □

Corollary 3.6 combines Theorem 3.1 and Corollary 3.3 from [9]. Note that the equivalence of (b) and (c) when $\theta = 0$ is Gallai's lemma for trees.

3.7 Lemma. *Let G be a connected graph. If $u \in V(G)$ and all paths in G starting at u are essential then G is critical.*

Proof. If all paths in $\mathcal{P}(u)$ are essential then Lemma 3.3 implies that all vertices in G are essential. Hence G is primitive, and it only remains for us to show that $\text{mult}(\theta, G) = 1$.

Let $T = T(G, u)$ be the path tree of G relative to u . From Theorem 2.2 we see that a path P from $\mathcal{P}(u)$ is essential in G if and only if it is essential in T . So our hypothesis implies that all paths in T which start at u are essential, whence Lemma 3.3 yields that all vertices in T are essential. Hence T is θ -primitive and therefore, by Corollary 3.6, θ is a simple zero of $\mu(T, x)$. Using Theorem 2.2 again we deduce that $\text{mult}(\theta, G) = 1$. \square

3.8 Lemma. *If u and v are essential vertices in G and v is not essential in $G \setminus u$ then there is a θ -essential path in $\mathcal{P}(u, v)$.*

Proof. Assume $\text{mult}(\theta, G) = k$. Our hypotheses imply that $\text{mult}(\theta, G \setminus uv) \geq k - 1$. If no path in $\mathcal{P}(u, v)$ is essential then, by Lemma 2.4, the multiplicity of θ as a zero of

$$\mu(G \setminus u, x) \mu(G \setminus v, x) - \mu(G, x) \mu(G \setminus uv, x)$$

is at least $2k$. Since θ has multiplicity $2k - 1$ as a zero of $\mu(G, x) \mu(G \setminus uv, x)$ it must also have multiplicity at least $2k - 1$ as a zero of $\mu(G \setminus u, x) \mu(G \setminus v, x)$. Hence u and v cannot both be essential. \square

If u and v are essential in G then v is essential in $G \setminus u$ if and only if u is essential in $G \setminus v$. Thus the hypothesis of Lemma 3.8 is symmetric in u and v , despite appearances.

3.9 Corollary. *Let G be a tree, let θ be a zero of $\mu(G, x)$ and let u be a vertex in G . Then all paths in $\mathcal{P}(u)$ are essential if and only if all vertices in G are essential.*

Proof. It follows from Lemma 3.3 that if all paths in $\mathcal{P}(u)$ are essential then all vertices in G are essential. Suppose conversely that all vertices in G are essential. By Corollary 3.6 it follows that $\text{mult}(\theta, G) = 1$. Hence the hypotheses of Lemma 3.8 are satisfied by any two vertices in G , and so any two vertices are joined by an essential path. Since G is a tree the path joining any two vertices is unique and therefore all paths in $\mathcal{P}(u)$ are essential. \square

4. Structure Theorems

We now apply the machinery we have developed in the previous section.

4.1 Lemma (De Caen [2]). *Let u and v be adjacent vertices in a bipartite graph. If u is 0-essential then v is 0-special.*

Proof. Suppose that u and v are 0-essential neighbours in the bipartite graph G . As uv is a path, using Corollary 2.5 we find that

$$\text{mult}(0, G \setminus uv) \geq \text{mult}(0, G) - 1 = \text{mult}(0, G \setminus u),$$

and therefore v is not essential in $G \setminus u$. It follows from Lemma 3.8 that there is a 0-essential path P in G joining u to v .

We now show that P must have even length. From this it will follow that P together with the edge uv forms an odd cycle, which is impossible. From the definition of the matchings polynomial we see that $\text{mult}(0, H)$ and $|V(H)|$ have the same parity for any graph H . As

$$\text{mult}(0, G \setminus P) = \text{mult}(0, G) - 1$$

we deduce that $|V(G)|$ and $|V(G \setminus P)|$ have different parity and therefore P has even length. □

In the above proof we showed that a 0-essential path in a graph must have even length. Consequently no edge, viewed as a path of length one, can ever be 0-essential. It follows that K_1 is the only connected graph such that all paths are 0-essential. In general any graph which is minimal subject to its matchings polynomial having a particular zero θ will have the property that all its paths are θ -essential.

Lemma 4.1 is not hard to prove without reference to the matchings polynomial. Note that it implies that in any bipartite graph there is a vertex which is covered by every maximal matching, and consequently that a bipartite graph with at least one edge cannot be 0-primitive. As noted by de Caen [2], this leads to a very simple inductive proof of König's lemma.

Our next result is a partial analog to the Edmonds-Gallai structure theorem. See, e.g., [8: Chapter 3.2].

4.2 Theorem. *Let θ be a zero of $\mu(G, x)$ with non-zero multiplicity k and let a be a positive vertex in G . Then:*

- (a) *if u is essential in G then it is essential in $G \setminus a$;*
- (b) *if u is positive in G then it is essential or positive in $G \setminus a$;*
- (c) *if u is neutral in G then it is essential or neutral in $G \setminus a$.*

Proof. If $\text{mult}(\theta, G \setminus u) = k - 1$ and $\text{mult}(\theta, G \setminus a) = k + 1$, it follows by interlacing that $\text{mult}(\theta, G \setminus au) = k$. Hence u is essential in $G \setminus a$. Now suppose that u is positive in G . If $\text{mult}(\theta, G \setminus au) \geq k + 1$ then θ has multiplicity at least $2k + 1$ as a zero of $p(x)$ where

$$p(x) := \mu(G \setminus u, x) \mu(G \setminus a, x) - \mu(G, x) \mu(G \setminus au, x). \quad (4.1)$$

By Lemma 2.4, the multiplicity of θ as a zero of $p(x)$ must be even. It follows that this multiplicity must be at least $2k + 2$ and hence that θ has multiplicity at least $2k + 2$ as a zero of $\mu(G, x) \mu(G \setminus au, x)$. Therefore $\text{mult}(\theta, G \setminus au) \geq k + 2$ and so, by interlacing, $\text{mult}(\theta, G \setminus au) = k + 2$ and u is positive in $G \setminus a$. If $\text{mult}(\theta, G \setminus ua) = k + 2$ and u is neutral in G , then the multiplicity of θ as a zero of $p(x)$ is at least $2k + 1$ and therefore at least $2k + 2$, but this implies that θ is a zero of $\mu(G \setminus u, x) \mu(G \setminus a, x)$ with multiplicity at least $2k + 2$. Thus we conclude that u is neutral or essential in $G \setminus a$. \square

We note that Theorem 4.2(a) holds even if a is only neutral. If a is neutral and u is essential in G but not in $G \setminus a$ then θ has multiplicity at least $2k - 1$ as a zero of (4.1) and so must have multiplicity at least $2k$ as a zero of $\mu(G, x) \mu(G \setminus au, x)$. Hence its multiplicity as a zero of $\mu(G \setminus u, x) \mu(G \setminus a, x)$ is at least $2k$, which is impossible.

The following consequence of Theorem 4.2 and the previous remark was proved for trees by Neumaier. (See [9: Theorem 3.4(iii)].)

4.3 Corollary. *Any special vertex is positive.*

Proof. Suppose that a is special in G , and that u is a neighbour of a which is essential in G . By part (a) of the theorem and the remark above, u is essential in $G \setminus a$ and therefore, by Lemma 3.4, a is positive in G . \square

Lemma 3.7 implies that if G is not θ -critical then it contains a path, P say, that is not essential. If we delete P from G then the multiplicity of θ as a zero of $\mu(G, x)$ cannot decrease. Hence we may successively delete ‘inessential’ paths from G , to obtain a graph H such that $\text{mult}(\theta, H) \geq \text{mult}(\theta, G)$ and all paths in H are essential. If $k = \text{mult}(\theta, H)$ then, by Lemma 3.7 again, H contains exactly k critical components. The following result is a sharpening of this observation, since it implies that if $\text{mult}(\theta, G) = k$ we may produce a graph with k critical components by deleting k vertex disjoint paths from G ,

4.4 Lemma. *Let G be a graph, let θ be a zero of $\mu(G, x)$ and let u be a θ -essential vertex of G . Suppose that there is a path in $\mathcal{P}(u)$ which is not θ -essential. Then there is a path P in G starting at u such that $\text{mult}(\theta, G \setminus P) = \text{mult}(\theta, G)$ and some component C of $G \setminus P$ is critical. All vertices of C are essential in G .*

Proof. Suppose that there are paths in $\mathcal{P}(u)$ which are not essential, choose one of minimum length and call it P . Let v be the end-vertex of P other than u and let P' be the path $P \setminus v$. Then P' is essential, hence

$$\text{mult}(\theta, G \setminus P') = \text{mult}(\theta, G) - 1$$

and, as P is not essential,

$$\text{mult}(\theta, G \setminus P) \geq \text{mult}(\theta, G).$$

But we get $G \setminus P$ from $G \setminus P'$ by deleting the single vertex v , therefore $\text{mult}(\theta, G \setminus P) = \text{mult}(\theta, G)$ and v is positive in $G \setminus P'$. Consequently, by Lemma 3.4, there is an essential vertex u_1 adjacent to v in $(G \setminus P') \setminus v = G \setminus P$.

We now prove by induction on the number of vertices that, if the conditions of the lemma hold, then there is a path P and a component C of $G \setminus P$ as claimed and, further, there is a vertex w in C adjacent to the end-vertex of P distinct from u such that all paths in C that start at w are essential in C .

Let H denote $G \setminus P$. If all paths in H starting at u_1 are essential then, by Lemma 3.7, the component C of H that contains u_1 is critical. If Q is a path in C starting at u_1 then $\text{mult}(\theta, C \setminus Q) < \text{mult}(\theta, C)$; this implies that the path formed by the concatenation of P and Q is essential in G and hence, by Lemma 3.3, that all vertices in C are essential in G .

Thus we may suppose that there is a path in H starting at u_1 that is not essential. Because H has fewer vertices than G , we may assume inductively that there is a path Q in H starting at u_1 such that $\text{mult}(\theta, H) = \text{mult}(\theta, H \setminus Q)$ and a critical component C of $H \setminus Q$ that contains a neighbour w of the end-vertex of Q distinct from u_1 . Further all the paths in C that start at w are essential.

Let PQ denote the path formed by concatenating P and Q . Then all claims of the lemma hold for G , PQ , u and C . □

The two results which follow provide a strengthening of the observation that the zeros of the matchings polynomial of a graph with a Hamilton path are simple.

4.5 Lemma. *Suppose that u and v are adjacent vertices in G such that $\mu(G \setminus u, x)$ and $\mu(G \setminus uv, x)$ have no common zero. Then $\mu(G, x)$ and $\mu(G \setminus u, x)$ have no common zero, and therefore both polynomials have only simple zeros.*

Proof. Assume by way of contradiction that θ is a common zero of $\mu(G, x)$ and $\mu(G \setminus u, x)$. If $\text{mult}(\theta, G) > 1$ then by Corollary 2.5 we see that θ is a zero of $\mu(G \setminus u, x)$ and $\mu(G \setminus uv, x)$. If $\text{mult}(\theta, G \setminus u) > 1$ then $\text{mult}(\theta, G \setminus uv) > 0$, by interlacing. Hence

$$\text{mult}(\theta, G) = \text{mult}(\theta, G \setminus u) = 1$$

and so u is a neutral vertex in G . It follows from Lemma 3.4 that no neighbour of u can be essential in $G \setminus u$ and consequently $\text{mult}(\theta, G \setminus uv) > 0$. \square

A simple induction argument on the length of P yields the following.

4.6 Corollary. *Let H be an induced subgraph of G and suppose that there is a vertex u in H and a path P in G such that*

$$V(H) \cap V(P) = u, \quad V(H) \cup V(P) = V(G).$$

If $\mu(H, x)$ and $\mu(H \setminus u, x)$ have no common zero then all zeros of G are simple. \square

Note that the path P in this corollary does not have to be an induced path. One consequence of it is that if a graph has a Hamilton path then the zeros of its matchings polynomial are all simple. However this result shows that there will be many other graphs with all zeros simple.

5. Eigenvectors

Let G be a graph with adjacency matrix $A = A(G)$. We view an eigenvector f of A with eigenvalue θ as a function on $V(G)$ such that

$$\theta f(u) = \sum_{i \sim u} f(i).$$

We denote the characteristic polynomial of G by $\phi(G, x)$. (It is defined to be $\det(xI - A(G))$.) We recall that for forests the characteristic and matchings polynomials are equal. Our first result follows from the proof of Theorem 5.2 in [3].

5.1 Lemma. *Let θ be an eigenvalue of the graph G and let u be a vertex in G . Then the maximum value of $f(u)^2$ as f ranges over the eigenvectors of G with eigenvalue θ and norm one is equal to $\phi(G \setminus u, \theta) / \phi'(G, \theta)$.* \square

5.2 Corollary (Neumaier [9: Theorem 3.4]). *Let T be a tree and let θ be a zero of its matchings polynomial. Then a vertex u is essential if and only if there is an eigenvector f of T such that $f(u) \neq 0$.* \square

5.3 Theorem. *Let T be a tree, let θ be a zero of $\mu(T, x)$ and let a be a vertex of T which is not essential. Then a vertex is essential in $T \setminus a$ if and only if it is essential in T . Further, if a is positive then it has an essential neighbour.*

Proof. Let W be the eigenspace of T belonging to θ and let W_a be the corresponding eigenspace of $T \setminus a$. Then W_a is the direct sum of the eigenspaces of the component of $T \setminus a$ belonging to θ and W is the subspace formed by the vectors f such that

$$\sum_{i \sim a} f(i) = 0.$$

If a is neutral then $W = W_a$ and so T and $T \setminus a$ have the same essential vertices. If a is positive then W is a proper subspace of W_a , whence it follows that there are vectors in W_a which are not zero on all neighbours of a . For each vector in W_a there is an eigenvector in W with the same support on $T \setminus a$. Hence a has an essential neighbour and any vertex which is essential in $T \setminus a$ is also essential in T . \square

Theorem 5.3 is a strengthening of a result of Neumaier [9: Corollary 3.5]. Suppose that T is a tree with exactly s special vertices and $\text{mult}(\theta, T) = k$. Then Theorem 5.3 together with Theorem 4.2 implies that we may successively delete the special vertices, obtaining a forest F with no special vertices and $\text{mult}(\theta, F) = k + s$. Hence any component of F is either θ -critical or does not have θ as a zero of its matchings polynomial. Therefore F has exactly $k + s$ θ -critical components, and these components form an induced subgraph of T .

6. Questions

Many problems remain. Here are some.

- (1) Must a positive vertex be special when $\theta \neq 0$? (If $\theta = 0$ then all vertices which are not essential are positive.)
- (2) What can be said of the graphs where every pair of vertices are joined by at least one essential path? (Or of the graphs with a vertex u such that all vertices can be joined to u by an essential path?)
- (3) Must a θ -primitive graph be θ -critical?

It might be interesting to investigate the case $\theta = 1$ in depth.

References

- [1] N. Biggs, *Algebraic Graph Theory*. (Cambridge U. P., Cambridge) 1974.
- [2] D. de Caen, On a theorem of König on bipartite graphs, *J. Comb. Inf. System Sci.* **13** (1988) 127.
- [3] C. D. Godsil, Matchings and walks in graphs, *J. Graph Theory*, **5**, (1981) 285–297.
- [4] C. D. Godsil and I. Gutman, On the theory of the matching polynomial, *J. Graph Theory*, **5** (1981), 137–144.
- [5] C. D. Godsil, Real graph polynomials, in *Progress in Graph Theory*, edited by J. A. Bondy and U. S. R. Murty, (Academic Press, Toronto) 1984, pp. 281–293.
- [6] C. D. Godsil, *Algebraic Combinatorics*. (Chapman and Hall, New York) 1993.
- [7] O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Commun. Math. Physics*, **25** (1972), 190–232.
- [8] L. Lovász and M. D. Plummer, *Matching Theory*. Annals Discrete Math. 29, (North-Holland, Amsterdam) 1986.
- [9] A. Neumaier, The second largest eigenvalue of a tree, *Linear Algebra Appl.* **48** (1982) 9–25.