

# A Plethysm Formula for $p_\mu(\underline{x}) \circ h_\lambda(\underline{x})$

William F. Doran IV

Department of Mathematics  
California Institute of Technology  
Pasadena, CA 91125

doran@cco.caltech.edu

Submitted: September 10, 1996; Accepted: May 2, 1997

## Abstract

This paper gives a new formula for the plethysm of power-sum symmetric functions and complete symmetric functions. The form of the main result is that for  $\mu \vdash b$  and  $\lambda \vdash a$  with length  $t$ , then

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_T \underline{\omega}^{\text{maj}_{\mu^t}(T)} s_{\text{sh}(T)}(\underline{x})$$

where the sum is over semistandard tableaux of weight  $\lambda_1^b \lambda_2^b \dots \lambda_t^b$  and  $\underline{\omega}^{\text{maj}_{\mu^t}(T)}$  is a root of unity which depends on  $\mu$ ,  $t$ , and  $T$ .

## 1. Introduction

This paper gives a formula for the plethysm of power-sum symmetric functions and complete symmetric functions. In Section 1, some tableaux and symmetric function notation is given. In Section 2, the work of [D] is reviewed. The key result in this section is that for  $\mu \vdash b$ ,

$$p_\mu(\underline{x}) \circ h_a(\underline{x}) = \sum_T \underline{\omega}^{\text{maj}_{\mu}(T)} s_{\text{sh}(T)}(\underline{x})$$

where the sum is over all semistandard tableaux  $T$  of weight  $a^b$  and  $\underline{\omega}^{\text{maj}_{\mu}(T)}$  is a root of unity which depends on  $T$  and  $\mu$ . In Section 3, a technical result which is needed later is proven. In Section 4, the main result of the paper is proven. The form of this result is that for  $\mu \vdash b$  and  $\lambda \vdash a$  with length  $t$

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_T \underline{\omega}^{\text{maj}_{\mu^t}(T)} s_{\text{sh}(T)}(\underline{x})$$

where the sum is over semistandard tableaux  $T$  of weight  $\lambda_1^b \lambda_2^b \dots \lambda_t^b$  and  $\underline{\omega}^{\text{maj}_{\mu^t}(T)}$  is a root of unity which depends on  $\mu$ ,  $t$ , and  $T$ .

---

Keywords: Symmetric Functions, Plethysm

Mathematics Subject Classification: Primary 05E05, Secondary 05E10

### 1.1 Tableaux

A *partition* of  $n$  is a weakly decreasing sequence  $\lambda = (\lambda_1 \geq \dots \geq \lambda_l)$  of positive integers which sum to  $n$ . Either  $|\lambda| = n$  or  $\lambda \vdash n$  is used to denote that  $\lambda$  is a partition of  $n$ . The value  $l$  is called the *length* of  $\lambda$  and is denoted  $l(\lambda)$ . Let  $[\lambda] = \{(i, j) : 1 \leq i \leq l(\lambda) \text{ and } 1 \leq j \leq \lambda_i\} \subset \mathbf{Z}^2$ . The set  $[\lambda]$  is the *Ferrers diagram* of  $\lambda$  and is thought of as a collection of boxes arranged using matrix coordinates.

A *composition* of  $n$  is a sequence  $\alpha = (\alpha_1, \dots, \alpha_l)$  of positive integers which sum to  $n$ . For notational purposes, let  $a_1^{b_1} a_2^{b_2} \dots a_l^{b_l}$  denote the composition

$$\underbrace{(a_1, \dots, a_1)}_{b_1 \text{ times}}, \underbrace{(a_2, \dots, a_2)}_{b_2 \text{ times}}, \dots, \underbrace{(a_l, \dots, a_l)}_{b_l \text{ times}}.$$

Also, given a composition  $\alpha = (\alpha_1, \dots, \alpha_l)$ , let  $\alpha^t$  denote the composition

$$(\alpha_1, \dots, \alpha_l, \dots, \alpha_1, \dots, \alpha_l)$$

where  $\alpha$  is repeated  $t$  times.

A *tableau* of *shape*  $\lambda$  and *weight* (or *content*)  $\alpha = (\alpha_1, \dots, \alpha_l)$  is a filling of the Ferrers diagram of  $\lambda$  with positive integers such that  $i$  appears  $\alpha_i$  times. A tableau is *semistandard* if its entries are weakly increasing from left to right in each row and strictly increasing down each column. The *Kostka number*  $K_{\lambda, \mu}$  equals the number of semistandard tableaux of shape  $\lambda$  and weight  $\mu$ . If  $[\nu] \subseteq [\lambda]$ , let  $[\lambda/\nu]$  denote the *skew-shape*  $[\lambda] \setminus [\nu]$ . A filling of  $[\lambda/\nu]$  with  $\alpha_i$   $i$ 's is a *skew-tableau* of shape  $\lambda/\nu$  and weight  $\alpha$ . A semistandard skew-tableau is defined similarly.

### 1.2 Symmetric Functions

The symmetric function notation in this papers closely follows that of Chapter 1 in Macdonald [M]. Let  $\Lambda$  denote the ring of symmetric functions with rational coefficients in the variables  $\{x_1, x_2, \dots\}$ . Let  $s_\lambda(\underline{x})$ ,  $p_\lambda(\underline{x})$ , and  $h_\lambda(\underline{x})$  denote the Schur symmetric functions, power-sum symmetric functions, and complete symmetric functions respectively. The ring  $\Lambda$  has a bilinear, symmetric, positive definite scalar product given by  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$ .

When two Schur symmetric functions are multiplied together and expanded in terms of Schur symmetric functions,

$$s_\mu(\underline{x})s_\nu(\underline{x}) = \sum_{\lambda \vdash |\mu|+|\nu|} c_{\mu, \nu}^\lambda s_\lambda(\underline{x}),$$

the resulting multiplication coefficients  $c_{\mu, \nu}^\lambda$  are nonnegative integers. These coefficients are called *Littlewood-Richardson coefficients*. See either Section I.9 of [M] or Section 4.9 of [S] for details.

Let  $f(\underline{x})$  and  $g(\underline{x})$  be symmetric functions. The *plethysm* of  $f(\underline{x})$  and  $g(\underline{x})$  is denoted  $f(\underline{x}) \circ g(\underline{x})$ . A definition of plethysm is given in Section I.8 of [M]. The plethysm results in this paper use a plethysm formula in [D] which is recalled in Section 2. Plethysm has the property of being algebraic in the first coordinate.

**Proposition 1.1.** *For all symmetric functions  $f_1(\underline{x})$ ,  $f_2(\underline{x})$ , and  $g(\underline{x})$  the following two identities hold.*

- (a)  $(f_1(\underline{x}) + f_2(\underline{x})) \circ g(\underline{x}) = (f_1(\underline{x}) \circ g(\underline{x})) + (f_2(\underline{x}) \circ g(\underline{x}))$ .
- (b)  $(f_1(\underline{x})f_2(\underline{x})) \circ g(\underline{x}) = (f_1(\underline{x}) \circ g(\underline{x}))(f_2(\underline{x}) \circ g(\underline{x}))$ .

While plethysm is not in general symmetric, the following is true.

**Proposition 1.2.** *For any symmetric function  $f(\underline{x})$  and positive integer  $n$ ,  $p_n(\underline{x}) \circ f(\underline{x}) = f(\underline{x}) \circ p_n(\underline{x})$ .*

Proofs of these propositions are given in Section I.8 of [M].

## 2. Formula for $p_\mu(\underline{x}) \circ h_a(\underline{x})$

This section reviews some results in [D].

**Definition:** Given a semistandard tableau or semistandard skew-tableau  $T$ ,  $i$  is a *descent* with *multiplicity*  $k$  if there exists  $k$  disjoint pairs  $\{(x_1, y_1), \dots, (x_k, y_k)\}$  of boxes in the Ferrers diagram of  $T$  such that the entry in each  $x_j$  is  $i$ , the entry in each  $y_j$  is  $i + 1$ ,  $y_j$  is in a lower row than  $x_j$  for all  $j$ , and there does not exist a set of  $k + 1$  pairs of boxes which satisfy these conditions. Let  $m_i(T)$  denote the multiplicity of  $i$  as a descent in  $T$ . Finally, the *Major index* of  $T$ , denoted  $\text{maj}(T)$ , is sum of the descents with multiplicity. That is,  $\text{maj}(T) = \sum i m_i(T)$ .

Given a semistandard tableau  $T$  whose weight has length  $b$  and a composition  $\alpha = (\alpha_1, \dots, \alpha_l)$  of  $b$ , decompose  $T$  into a sequence of semistandard skew-tableaux  $(T_1, \dots, T_l)$  by letting the shape of  $T_1$  be those positions in  $T$  which contain 1 through  $\alpha_1$ ,  $T_2$  be those positions in  $T$  which contain  $\alpha_1 + 1$  through  $\alpha_2$ , and so on. The actual entries in  $T_i$  are the entries of  $T$  but reindexed so they run from 1 to  $\alpha_i$ .

**Definition:** Following the notation of the previous paragraph, define the root of unity

$$\underline{\omega}^{\text{maj}_\alpha(T)} = \prod_{i=1}^l \omega_{\alpha_i}^{\text{maj}(T_i)}$$

where the  $\omega_k = e^{2\pi i/k}$ .

**Example 1:** Consider

$$T = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & & \end{array}$$

$T$  is a semistandard tableau of shape  $(5, 5, 3)$  and weight  $(3, 4, 3, 2, 1)$ . The descent multiplicities are  $m_1(T) = 3$ ,  $m_2(T) = 2$ ,  $m_3(T) = 2$ ,  $m_4(T) = 1$ . Its Major index is 17. Take  $\alpha = (2, 3)$ . Then the decomposition of  $T$  is

$$T_1 = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & \end{array} \quad \text{and} \quad T_2 = \begin{array}{cccc} & & & 1 \\ & & 1 & 2 \\ 1 & 2 & 3 & \end{array}$$

The Major indices are  $\text{maj}(T_1) = 3$  and  $\text{maj}(T_2) = 4$ . Finally,  $\underline{\omega}^{\text{maj}_\alpha(T)} = \omega_2^3 \omega_3^4 = -\omega_3$ .

The main result of [D] is the following.

**Theorem 2.1.** *Let  $\mu \vdash b$ , then*

$$p_\mu(\underline{x}) \circ h_a(\underline{x}) = \sum_{\substack{\text{SST } T \\ \text{wt}(T) = a^b}} \underline{\omega}^{\text{maj}_\mu(T)} s_{\text{sh}(T)}(\underline{x}).$$

Here is a quick application of this formula. It is a generalization of Example 9(b) in Section I.8 of [M].

**Proposition 2.2.** *If  $b$  is prime power  $p^n$ , then  $\langle p_b(\underline{x}) \circ h_a(\underline{x}), s_\lambda(\underline{x}) \rangle \equiv K_{\lambda, a^b} \pmod{p}$ .*

**Proof:** By Theorem 2.1,  $\langle p_b(\underline{x}) \circ h_a(\underline{x}), s_\lambda(\underline{x}) \rangle$  is a sum of  $K_{\lambda, a^b}$  many terms where each term is a  $p^n$ -th root of unity.

One last result needed (Theorem 2.1 of [D]). This concerns an algorithm for selecting the  $(x_j, y_j)$  pairs which contribute to the statistic  $m_i(T)$  for a given  $T$ . Here is the algorithm.

```

Select an ordering  $\sigma$  of the  $\alpha$  many boxes which contain an  $i$ .
Set  $j = 0$ .
For  $k = 1$  to  $\alpha$  Do
  Let  $x = \sigma(k)$ .
  If there are any boxes below  $x$  which contain  $i + 1$  and have not already been
  selected as an  $y_j$  Then
    Increment  $j$  by 1.
    Let  $x_j = x$ .
    Let  $y_j$  be the right-most of these unselected boxes containing  $i + 1$ .
  End If
End Do
    
```

Algorithm 1.



It is clear that  $J(T)$  is semistandard and that  $\text{wt}(J(T)) = \text{wt}(T)$ . However it is not clear from the definition that this operation is well-defined. The choices for the initial position of  $F$  in each iteration might effect the outcome of the algorithm. But, it has been shown (see Section 3.9 of [S] for a proof) that for all choices of the  $F$ 's, the resulting  $J(T)$  is the same. The next result gives an important property of the Jeu de Taquin operation. A proof is given in the proof of Theorem 4.9.4 of [S].

**Lemma 3.1.** *Given a semistandard tableau  $Q$  of shape  $\eta$ , the number of semistandard tableaux  $T$  of shape  $\lambda/\nu$  such that  $J(T) = Q$  equals the Littlewood-Richardson coefficient  $c_{\nu,\eta}^\lambda$ .*

Another fact about Jeu-de-Taquin is that preserves descent multiplicities.

**Lemma 3.2.** *Let  $T$  be a semistandard tableau whose weight has length  $b$  and let  $\alpha$  be a composition of  $b$ . Then  $m_i(T) = m_i(J(T))$  for all  $i$ .*

**Proof:** Suppose  $T \mapsto T'$  in one application of the inner loop of the Jeu de Taquin algorithm. We want to show that  $m_i(T) = m_i(T')$  for all  $i$ . If the swap of this kind

$$\begin{array}{ccc} F & j & \\ k & & \end{array} \rightarrow \begin{array}{ccc} j & F & \\ & k & \end{array},$$

where  $k > j$ , then since no element has changed which row it occupies, it is clear that all  $m_i(T)$  are unchanged. However, if the swap is of the other kind

$$\begin{array}{ccc} F & j & \\ & & k \end{array} \rightarrow \begin{array}{ccc} & & k \\ k & & F \end{array} j$$

where  $k \leq j$ , then there are two possible problems: some  $(k, k + 1)$  pairs may have been created, or some  $(k - 1, k)$  pairs may have been destroyed. Either of these “problems” might alter the value of  $m_k(T)$  or  $m_{k-1}(T)$ , respectively.

First, we show that any new  $(k, k + 1)$  pairs do not effect the value of  $m_k(T)$ . Let  $r$  be the row containing  $F$  in  $T$ . So, the possible new  $(k, k + 1)$  pairs in  $T'$  are from the  $k$  which just moved into row  $r$  and the  $k + 1$ 's in row  $r + 1$ . Since  $T$  and  $T'$  are semistandard, each of these  $k + 1$  must have a  $k$  directly above them in both  $T$  and  $T'$ . The key is apply Theorem 2.3 with the properly chosen ordering  $\sigma$  of the  $k$ 's. The proper choice is to place the  $k$ 's in row  $r$  which have a  $k + 1$  directly below them (say from right to left) at the beginning of  $\sigma$ , then fill in all the other  $k$ 's (again say from right to left). Now apply the algorithm to determine  $m_k(T)$  and  $m_k(T')$ . In both cases, the first group of  $k$ 's in row  $r$  with a  $k + 1$  below them get

matched with the  $k + 1$  directly below them. Thus, all of the new  $(k, k + 1)$  pairs in  $T'$  do effect the outcome of the algorithm since all of the  $k + 1$ 's get selected before the algorithm can consider using one of these pairs. Therefore,  $m_k(T) = m_k(T')$ .

The proof that  $m_{k-1}(T) = m_{k-1}(T')$  is similar. The  $(k - 1, k)$  pairs that get destroyed in  $T'$  are from the  $k$  which moved into row  $r$  and the  $k - 1$ 's in row  $r$ . Since  $T$  is semistandard, each of the  $k - 1$ 's in row  $r$  must have a  $k$  directly below them in both  $T$  and  $T'$ . Order the  $k - 1$ 's by placing the  $k - 1$ 's in row  $r$  first and other  $k - 1$ 's later. The algorithm for determining  $m_{k-1}(T)$  and  $m_{k-1}(T')$  give the same result.

Finally, these two lemmas are put together to prove a key technical result used in the next section.

**Theorem 3.3.** *Let  $|\lambda/\nu| = n$ ,  $\beta$  be a composition of  $n$  with  $l$  parts, and  $\alpha$  be a composition of  $l$ . Then*

$$\sum_{\substack{\text{SSST } T \\ \text{sh}(T)=\lambda/\nu, \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} = \sum_{\eta \vdash n} \left( c_{\nu, \eta}^\lambda \sum_{\substack{\text{SST } T \\ \text{sh}(T)=\eta, \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} \right).$$

**Proof:** By Lemma 3.1, Jeu-de-Taquin provides the bijection between the two sides. By Lemma 3.2,  $\underline{\omega}^{\text{maj}_\alpha(T)}$  is preserved under this bijection.

#### 4. Formula for $p_\mu(\underline{x}) \circ h_\lambda(\underline{x})$

**Theorem 4.1.** *Given  $\mu \vdash b$  and  $\lambda \vdash a$  where the length of  $\mu$  is  $l$  and the length of  $\lambda$  is  $t$ , let  $\alpha = \mu^t$  and  $\beta = \lambda_1^b \lambda_2^b \dots \lambda_t^b$ . So,  $\beta$  is a composition of  $ab$  with  $tb$  parts, and  $\alpha$  is a composition of  $tb$ . Then*

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_{\substack{\text{SST } T \\ \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} s_{\text{sh}(T)}(\underline{x}).$$

**Proof:** Proof by induction on  $t$ . The case  $t = 1$  is Theorem 2.1. So assume the theorem is valid when  $\mu$  has length less than  $t$ . Let  $\lambda^* = (\lambda_1, \dots, \lambda_{t-1})$ . Now a computation gives the desired result. Explanations of the steps are given after the computation.

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \prod_i p_{\mu_i}(\underline{x}) \circ h_{\lambda_i}(\underline{x}) \tag{1}$$

$$= \prod_i h_{\lambda}(\underline{x}) \circ p_{\mu_i}(\underline{x}) \tag{2}$$

$$= \prod_i (h_{\lambda^*}(\underline{x}) \circ p_{\mu_i}(\underline{x})) (h_{\lambda_t}(\underline{x}) \circ p_{\mu_i}(\underline{x})) \tag{3}$$

$$= \prod_i (p_{\mu_i}(\underline{x}) \circ h_{\lambda^*}(\underline{x})) \prod_i (p_{\mu_i}(\underline{x}) \circ h_{\lambda_t}(\underline{x})) \tag{4}$$

$$= \left( \sum_{\substack{\text{SST } T_1 \\ \text{wt}(T_1)=\lambda_1^b \dots \lambda_{t-1}^b}} \underline{\omega}^{\text{maj}_{\mu^{t-1}}(T_1)} s_{\text{sh}(T_1)}(\underline{x}) \right) \left( \sum_{\substack{\text{SST } T_2 \\ \text{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\text{maj}_{\mu}(T_2)} s_{\text{sh}(T_2)}(\underline{x}) \right) \tag{5}$$

$$= \sum_{\nu \vdash ab} s_{\nu}(\underline{x}) \left[ \sum_{\substack{\text{SST } T_1 \\ \text{wt}(T_1)=\lambda_1^b \dots \lambda_{t-1}^b}} \left( \underline{\omega}^{\text{maj}_{\mu^{t-1}}(T_1)} \left( c_{\text{sh}(T_1), \eta}^{\nu} \sum_{\substack{\text{SST } T_2 \\ \text{sh}(T_2)=\eta, \text{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\text{maj}_{\mu}(T_2)} \right) \right) \right] \tag{6}$$

$$= \sum_{\nu \vdash ab} s_{\nu}(\underline{x}) \left[ \sum_{\substack{\text{SST } T_1 \\ \text{wt}(T_1)=\lambda_1^b \dots \lambda_{t-1}^b}} \left( \underline{\omega}^{\text{maj}_{\mu^{t-1}}(T_1)} \left( \sum_{\substack{\text{SST } T_2 \\ \text{sh}(T_2)=\nu/\text{sh}(T_1), \text{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\text{maj}_{\mu}(T_2)} \right) \right) \right] \tag{7}$$

$$= \sum_{\nu \vdash ab} s_{\nu}(\underline{x}) \left[ \sum_{\substack{\text{SST } T_3 \\ \text{sh}(T_3)=\nu, \text{wt}(T_3)=\beta}} \underline{\omega}^{\text{maj}_{\alpha}(T_3)} \right] \tag{8}$$

(1) Apply Proposition 1.1(b). (2) Apply Proposition 1.2. (3) Apply Proposition 1.1(b). (4) Apply Proposition 1.2. (5) Apply the induction hypothesis. (6)

Multiply the Schur symmetric functions and collect the terms which contribute to  $s_\nu(\underline{x})$ . (7) Apply Theorem 3.3. (8) Combine  $T_1$  and  $T_2$  to form  $T_3$  by first reindexing the values in  $T_2$  to run from  $b(\lambda_1 + \cdots + \lambda_{t-1}) + 1$  to  $b(\lambda_1 + \cdots + \lambda_t)$  and then place  $T_1$  inside the reindexed  $T_2$ . Notice that under this bijection  $\underline{\omega}^{\text{maj}_{\mu^{t-1}}(T_1)} \underline{\omega}^{\text{maj}_\mu(T_2)} = \underline{\omega}^{\text{maj}_\alpha(T_3)}$ .

**Example 3:** Let  $\mu = (2)$  and  $\lambda = (2, 1)$ . Then  $\alpha = (2, 2)$  and  $\beta = (2, 2, 1, 1)$ . The semistandard tableaux  $T$  with weight  $\beta$  and  $\underline{\omega}^{\text{maj}_\alpha(T)}$  are given in Table 1. So,  $p_{(2)}(\underline{x}) \circ h_{(2,1)}(\underline{x}) = s_{(6)}(\underline{x}) - s_{(5,1)}(\underline{x}) + 2s_{(4,2)}(\underline{x}) - 2s_{(3,3)}(\underline{x}) - s_{(4,1,1)}(\underline{x}) + s_{(2,2,2)}(\underline{x}) + s_{(3,1,1,1)}(\underline{x}) - s_{(2,2,1,1)}(\underline{x})$ .

### References

- [D] W. Doran, A New plethysm Formula for Symmetric Functions, *J. of Alg. Combin.*, submitted.
- [M] I. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Clarendon Press, Oxford, 1995.
- [S] B. Sagan, "The Symmetric Group: Representations, Combinatorial Algorithms, & Symmetric Functions," Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991.

$T$	$\underline{\omega}^{\text{maj}_{(2,2)}(T)}$	$T$	$\underline{\omega}^{\text{maj}_{(2,2)}(T)}$
1 1 2 2 3 4	1	1 1 2 2 3 4	-1
1 1 2 2 4 3	1	1 1 2 4 2 3	-1
1 1 2 2 3 4	-1	1 1 2 3 2 4	1
1 1 2 3 4 2	-1	1 1 2 2 3 4	1
1 1 2 2 3 4	1	1 1 2 2 4 3	-1
1 1 2 4 2 3	-1	1 1 4 2 2 3	1
1 1 2 3 2 4	1	1 1 3 2 2 4	-1
1 1 3 4 2 2	1	1 1 2 2 3 4	1
1 1 2 2 3 4	1	1 1 2 2 3 4	1
1 1 3 2 2 4	1	1 1 2 2 3 4	-1

Table 1: Semistandard Tableaux of Weight (2, 2, 1, 1)