Structural Induction (cont)

COMP2600 — Formal Methods for Software Engineering

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Induction on Finite Trees

Like natural numbers and lists, trees are also *inductively defined*.

1. $Nul\ ::\ Tree\ \ a$
2. If $x :: a$ and $t_1 :: Tree\ a$ and $t_2 :: Tree\ a$
   then $Node\ \ x\ \ t_1\ \ t_2\ \ ::\ \ Tree\ \ a$

No object is a tree of a's unless justified by these clauses.

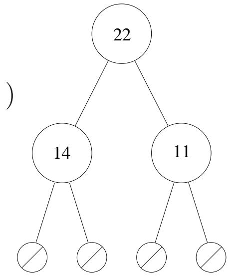
To prove a property for all trees (of objects of some type):

- Prove it for Nul
- Prove that, whenever it is true for t_1 and t_2 it is true for $Node\ \ x\ \ t_1\ \ t_2$

Why does it Work?

Suppose we have proved:

- The *base case*: $P(Nul)$
- The *step case*: $\forall t_1.\ \forall t_2.\ P(t_1) \wedge P(t_2) \rightarrow P(Node\ x\ t_1\ t_2\ )$



We can use these facts to prove
$P(Node\ 22\ (Node\ 14\ Nul\ Nul)\ (Node\ 11\ Nul\ Nul))$

1. $P(Nul)$ is given
2. $P(Node\ 14\ Nul\ Nul)$ follows from $P(Nul)$ and $P(Nul)$
3. $P(Node\ 11\ Nul\ Nul)$ follows from $P(Nul)$ and $P(Nul)$
4. $P(Node\ 22\ (Node\ 14\ Nul\ Nul)\ (Node\ 11\ Nul\ Nul))$
   follows from $P(Node\ 14\ Nul\ Nul)$ and $P(Node\ 11\ Nul\ Nul)$

That's Induction on Structure

The rule of *Structural Induction for Trees* is usually written as:

$$\frac{P(Nul) \qquad \forall x.\ \forall t_1.\ \forall t_2.\ P(t_1) \wedge P(t_2) \rightarrow P(Node\ x\ t_1\ t_2)}{\forall t.\ P(t)}$$

or, being fussy with types:

$$\frac{P(Nul :: Tree\ a) \quad \forall (x :: a).\forall (t_1\ t_2 :: Tree\ a).\ P(t_1) \wedge P(t_2) \rightarrow P(Node\ x\ t_1\ t_2)}{\forall (t :: Tree\ a).\ P(t)}$$

Standard functions

```
mapT f Nul              = Nul                    -- (M1)
mapT f (Node x t1 t2)
     = Node (f x) (mapT f t1) (mapT f t2) -- (M2)

count Nul               = 0                      -- (C1)
count (Node x t1 t2)
    = 1 + count t1 + count t2                    -- (C2)
```

The theorem we prove about trees is `count (mapT f t) = count t`

It is the tree analog of the list property `length (map f xs) = length xs`

`count (mapT f t) = count t`

This time we're doing induction over trees.
The smallest possible tree is Nul

Base Case: $P(Nul)$

```
count (mapT f Nul) = count Nul
```

This holds by `(M1)`

Step Case: $\forall x.\ \forall t_1.\ \forall t_2.\ P(t_1) \wedge P(t_2) \rightarrow P(Node\ x\ t_1\ t_2)$

This time the induction hypothesis is $P(u_1) \wedge P(u_2)$, but we will write both parts separately.

Assume $P(u_1) \wedge P(u_2)$:

```
count (mapT f u1) = count u1            -- (IH1)
count (mapT f u2) = count u2            -- (IH2)
```

Prove $P(Node\ a\ u_1\ u_2)$, that is,

```
count (mapT f (Node a u1 u2)) = count (Node a u1 u2)
```


Prove $P(Node\ a\ u_1\ u_2)$, that is,

```
count (mapT f (Node a u1 u2)) = count (Node a u1 u2)
```

```
count (mapT f (Node a u1 u2))
= count (Node (f x) (mapT f u1) (mapT f u2)) -- by (M2)
= 1 + count (mapT f u1) + count (mapT f u2)  -- by (C2)
= 1 + count u1 + count u2                   -- by (IH1, IH2)
= count (Node a u1 u2)                      -- by (C2)
```

Theorem proved!

Observe the Trilogy Again

There are three related stories exemplified here, now for trees

- **Inductive Definition**

```
data Tree a = Nul | Node a (Tree a) (Tree a)
```

- **Recursive Function Definitions**

```
f  Nul          = ...
f (Node x t1 t2) = ... f(t1) ... f(t2) ...
```

- **Structural Induction Principle**

  Prove $P(Nul)$

  Prove $\forall x. \forall t_1. \forall t_2.\ P(t_1) \wedge P(t_2) \rightarrow P(Node\ x\ t_1\ t_2)$

The similarity is that each has a base case and a step case.

The form of the inductive type definition determines the form of recursive function definitions and the structural induction principle.

Structural Induction *vs* accumulating parameters

Here are two versions of the sum function. The second one uses an accumulating parameter and requires less space at run-time.

```
sum1 []            = 0                    -- (S1)
sum1 (x:xs)        = x + sum1 xs          -- (S2)

sum2 xs            = sum2' 0 xs           -- (T1)
sum2' acc []       = acc                  -- (T2)
sum2' acc (x:xs)   = sum2' (acc + x) xs   -- (T3)
```


sum1 xs = sum2 xs

Try and prove that the two definitions of sum are equivalent:

Base Case: $P([])$

```
sum2 []    = sum1 []

sum2 [] = sum2' 0 []   -- by (T1)
        = 0            -- by (T2)
        = sum1 []      -- by (S1)
```


Step Case: $\forall x. \forall xs. P(xs) \rightarrow P(x : xs)$

Assume:

```
sum2 as      = sum1 as              -- (IH)
```

Prove:

```
sum2 (a:as) = sum1 (a:as)

sum2 (a:as) = sum2' 0 (a:as)       -- by (T1)
            = sum2' (0 + a) as     -- by (T3)

sum1 (a:as) = a + sum1 as          -- by (S2)
            = a + sum2 as              -- by (IH)
            = a + sum2' 0 as           -- by (T1)
```

Now we're stuck. We used the IH, but both sides aren't the same...

Proving a Stronger Property

Sometimes we need to prove a stronger property than the one we are given.

How to describe `sum2'` ? `sum2' acc xs` is `acc` + sum of `xs`

So here is a property which involves the current accumulator in `sum2'`

Let $P(xs)$ be ∀`acc. acc + sum1 xs = sum2' acc xs`

Why include the ∀`acc.` ? (We haven't done this before) — see next slide

Base Case: $P([\,])$ ∀`acc. acc + sum1 [] = sum2' acc []`

```
acc + sum1 [] = acc + 0 = acc  -- by (S1)
              = sum2' acc []   -- by (T2)
```

Then generalise over `acc` — uses ∀-I

Step Case: $\forall x.\forall xs.P(xs) \to P(x:xs)$

Assume: ∀`acc. acc + sum1 as = sum2' acc as   -- (IH)`

Prove: ∀`acc. acc + sum1 (a:as) = sum2' acc (a:as)`

```
acc + sum1 (a:as) = acc + a + sum1 as   -- by (S2)
                  = sum2' (acc + a) as  -- by (IH) (*)
                  = sum2' acc (a:as)    -- by (T3)
```

Then generalise over `acc` — uses ∀-I.
Note also the use of associativity of $+$

How is `(*)` a use of the induction hypothesis ?
Now, our induction hypothesis is ∀`acc. ...`
We can instantiate it at `(acc + a)` to give the equality we need.
Without ∀`acc` in the induction hypothesis, this proof would not work.

Strong(xs) → Weak(xs)

We have now proved $\forall xs.\, P(xs)$, that is:

∀`xs.` ∀`acc. acc + sum1 xs = sum2' acc xs`

Change the order of the quantifiers:

∀`acc.` ∀`xs. acc + sum1 xs = sum2' acc xs`

Instantiate at `(acc = 0)`

```
∀xs. 0 + sum1 xs = sum2' 0 xs    -- by ∀-E
∀xs. sum1 xs     = sum2' 0 xs    -- by arith
∀xs. sum1 xs     = sum2 xs       -- by T1
```

Which was our original property required.

When might a stronger property P be necessary ?

The clue here is that

- To evaluate `sum2 xs` , that is `sum2' 0 xs` , there are recursive steps of evaluating `sum2' acc xs` , for `acc` $\neq 0$.
- Likewise, to prove something about `sum2 xs` , that is `sum2' 0 xs` , there are inductive steps where you prove something about `sum2' acc xs` , for `acc` $\neq 0$

To put it another way

- in the code we need to define a function (`sum2'` , works for *all* `acc`) which is *more* capable than we want (`sum2` , uses only `acc` $= 0$)
- in the proof we need to prove a *stronger* result (for *all* `acc`) than we want (for `acc` $= 0$)

Look at proving it for $xs = [2,3,5]$

We can write out a backwards form of the proof:

```
0 + sum1 [2,3,5] = sum2' 0 [2,3,5]   because
0 + 2 + sum1 [3,5] = sum2' (0+2) [3,5]   because
0 + 2 + 3 + sum1 [5] = sum2' (0+2+3) [5]   because
0 + 2 + 3 + 5 + sum1 [] = sum2' (0+2+3+5) []   because
0 + 2 + 3 + 5 = (0+2+3+5)
```

This "proof attempt" terminates because the list gets smaller each time, even though the accumulator changes (in fact it gets bigger).

In the same way, in the function definition for `sum2'`, evaluation of `sum2' acc [2,3,5]` terminates because the list gets smaller each time, even though the accumulator changes (in fact it gets bigger).

Another example

```
flatten :: Tree a -> [a]
flatten Nul            = []                                  -- (F1)
flatten (Node a t1 t2) = flatten t1 ++ [a] ++ flatten t2    -- (F2)

flatten2 :: Tree a -> [a]
flatten2 tree = flatten2' tree []                            -- (G)
flatten2' :: Tree a -> [a] -> [a]
flatten2' Nul acc = acc                                      -- (H1)
flatten2' (Node a t1 t2) acc =
    flatten2' t1 (a : flatten2' t2 acc)                      -- (H2)
```

Prove, by induction on the structure of binary trees, that for all `t :: Tree a`, and for all `acc :: [a]`,

```
flatten2' t acc = flatten t ++ acc
```


The property P we prove by induction will be

`P(t) = ∀acc.  flatten2' t acc = flatten t ++ acc`

Base case: `t = Nul`.
Show that `flatten2' Nul acc = flatten Nul ++ acc`

```
flatten2' Nul acc = acc                  -- by (H1)
       = [] ++ acc                       -- by (A1)
       = flatten Nul ++ acc              -- by (F1)
```

Step Case: `t = Node y t1 t2`.
Show that if, *for all* `acc`

```
flatten2' t1 acc = flatten t1 ++ acc       -- (IH1)
flatten2' t2 acc = flatten t2 ++ acc       -- (IH2)
```

then, *for all* `acc`
`flatten2' (Node y t1 t2) acc = flatten (Node y t1 t2) ++ acc`

Proof: Let a be given (we will generalise a to ∀`acc`)

```
flatten2' (Node y t1 t2) a
  = flatten2' t1 (y : flatten2' t2 a)         -- by (H2)
  = flatten t1 ++ (y : flatten2' t2 a)        -- (IH1)(*)
  = flatten t1 ++ (y : flatten t2 ++ a)       -- (IH2)(*)
  = flatten t1 ++ ((y : flatten t2) ++ a)     -- by (A2)
  = (flatten t1 ++ (y : flatten t2)) ++ a     -- (++ assoc)
  = flatten (Node y t1 t2) ++ a               -- by (F2)
```

(∗) Note - the acc in (IH1) is instantiated to `(y : flatten2' t2 a)`,
the acc in (IH2) is instantiated to a

Then we can generalise (using the ∀-I rule) a to acc, to get
∀`acc. flatten2' (Node y t1 t2) acc = flatten (Node y t1 t2) ++ acc`
which completes the proof.