Soundness and Completeness

COMP2600 — Formal Methods for Software Engineering

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Introduction

Various proof methods to prove a statement in propositional calculus

- Truth tables
- the Natural Deduction rules
- Simplification by algebraic (equivalence) rules

We hope these all are correct, whatever that means.

At least we would hope these all give the same answers!

Correctness

We *define* correctness (validity) by reference to truth tables.

Statement to be tested: $(p \wedge (q \vee r)) \rightarrow (q \vee p)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$q \vee p$	$(p \wedge (q \vee r)) \rightarrow (q \vee p)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	F	T
F	F	F	F	F	F	T

We regard it as correct, or *valid*, because truth-tables show it is true for *all* possible truth-values of p, q, r

Validity of a rule

A similar table shows that the rule $\dfrac{p \wedge (q \vee r)}{q \vee p}$ is valid, because whenever the premise is true, so is the conclusion

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$q \vee p$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	F
F	F	F	F	F	F

Soundness and Completeness

We say that a proof system (such as the rules of Natural Deduction) is

Sound, if every derivable (ie, provable) statement or rule is valid

Complete, if every valid statement or rule is derivable

Example: the rule $\dfrac{p \quad q}{p \wedge q}$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Whenever both the premises are true, then so is the conclusion, so that rule is sound.

Soundness of the $\vee$-E rule

$$\dfrac{p \vee q \qquad \begin{array}{c}[p]\\ \vdots \\ r\end{array} \qquad \begin{array}{c}[q]\\ \vdots \\ r\end{array}}{r}$$

There will be a sort of inductive argument here: to prove that the whole proof using this rule is valid, we assume that the sub-proofs are valid.

Soundness of the $\vee$-E rule

p	q	r	$p \vee q$	$\frac{[p]}{r}$	$\frac{[q]}{r}$	r
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Whenever all the "premises" (sub-proofs) are true (valid), then the conclusion is true, so the $\vee$-E rule is valid

Soundness — summary

- We have shown that some of the Natural Deduction rules are sound
- Likewise all the other rules are sound
- So, whenever the premises (if any) of a proof are true, then each subsequent line of the proof will be true (the rules "preserve" truth)
- so the conclusion will be true
- formally, this is a doubly-inductive argument over the length of the proof
  - assume subproofs within the proof are valid
  - assume previous lines at current indentation level are true

So the formula or rule produced by a natural deduction proof will be valid

Completeness

- Completeness is a property of the whole set of rules, rather than of individual rules.
- Completeness says that any valid formula or rule can be proved by the rules of Natural Deduction

How to do this?

- Just reproduce the truth-table in a natural deduction proof
- Uses the Excluded Middle Lemma, stated in lectures, $p \vee \neg p$
- Contains nested uses of the $\vee$-E rule
- It is rather long — you prove the same conclusion 2^n times
- Steps of each sub-proof similar to the calculations in the truth-table

1	$p \vee \neg p$	Lemma
2	$q \vee \neg q$	Lemma
3	│ p	
4	│ │ q	
5	│ │ ⋮	
6	│ │ c	
7	│ │ $\neg q$	
8	│ │ ⋮	
9	│ │ c	
10	│ c	$\vee$-E, 2, 4–6, 7–9
11	│ $\neg p$	
12	│ │ q	
13	│ │ ⋮	
14	│ │ c	
15	│ │ $\neg q$	
16	│ │ ⋮	
17	│ │ c	
18	│ c	$\vee$-E, 2, 12–14, 15–17
19	c	$\vee$-E, 1, 3–10, 11–18

How do do a proof which imitates the truth table calculation

The subproofs above imitate the calculations of each row of the truth table.

A row of the truth table involves several "calculations", using the truth table rules.

We can imitate these calculations by a proof, using similar steps.

One step in calculating the truth table corresponds to one use of a rule or derived rule, such as those on the next slides.

Completeness – proving truth-table values for $\to$-I

p	q	$p \to q$	corresponding "rule" to prove
T	T	T	$\dfrac{p \quad q}{p \to q}$
T	F	F	$\dfrac{p \quad \neg q}{\neg(p \to q)}$
F	T	T	$\dfrac{\neg p \quad q}{p \to q}$
F	F	T	$\dfrac{\neg p \quad \neg q}{p \to q}$

Completeness – proving truth-table values for $\wedge$-I

p	q	$p \wedge q$	corresponding "rule" to prove
T	T	T	$\dfrac{p \qquad q}{p \wedge q}$
T	F	F	$\dfrac{p \qquad \neg q}{\neg(p \wedge q)}$
F	T	F	$\dfrac{\neg p \qquad q}{\neg(p \wedge q)}$
F	F	F	$\dfrac{\neg p \qquad \neg q}{\neg(p \wedge q)}$

Derived Rules to imitate calculating a truth table row

You already have rules like $\dfrac{p \qquad q}{p \wedge q}$ and $\dfrac{p}{p \vee q}$

You need rules like $\dfrac{\neg p \qquad q}{\neg(p \wedge q)}$ and $\dfrac{\neg p \qquad \neg q}{\neg(p \vee q)}$

1	$\neg p$	
2	$\mid\ p \wedge q$	
3	$\mid\ p$	$\wedge$-E, 2
4	$\mid\ p \wedge \neg p$	$\wedge$-I, 3, 1
5	$\neg(p \wedge q)$	$\neg$I, 2–4

(premise q is not actually needed)

1	$\neg p$	
2	$\neg q$	
3	$\mid\ p \vee q$	
4	$\mid\ \mid\ p$	
5	$\mid\ \mid\ p \wedge \neg p$	$\wedge$-I, 4, 1
6	$\mid\ \mid\ q$	
7	$\mid\ \mid\ q \wedge \neg q$	$\wedge$-I, 6, 2
8	$\mid\ \mid\ p \wedge \neg p$	Lemma
9	$\mid\ p \wedge \neg p$	$\vee$-E, 3, 4–5, 6–8
10	$\neg(p \vee q)$	$\neg$I, 3–9

The Lemma is Contradiction Elimination (lectures), that from a contradiction such as $q \wedge \neg q$ you can prove anything, such as $p \wedge \neg p$

Consistency

- A logical system is *consistent* if it cannot prove falsity, ie (in our system) contradiction, $p \wedge \neg p$.
- In Natural Deduction (as in most logical systems), if you can prove $p \wedge \neg p$ then you can prove anything. So what's the use of that ?
- An alternative definition of consistency is that in an inconsistent logical system you can prove *anything*
- This definition is equivalent to the first one if, in that system, you can prove anything from a contradiction

Soundness and Completeness in Predicate Calculus

Recall the Propositional Calculus

- For the Propositional Calculus we defined correctness / validity by truth-tables.
- A statement or rule is *valid* if shown by truth-tables; that is, you look at *all* possible (truth-)values of the propositional variables $p, q, r, \ldots$ involved
- In this way you can *determine* whether a statement or rule is valid or not (Propositional Calculus is *decidable*).
- A choice of truth-values for $p, q, r, \ldots$ (ie, a row of the truth-table) is called an *interpretation*

Soundness and Completeness in Predicate Calculus

The Predicate Calculus is similar but more complicated.

- Imagine a set (domain) D of individuals, $a, b, c, \ldots$
- If the formulae involve predicates $P, Q, R, \ldots$, there are propositions $P(a), P(b), P(c), \ldots, Q(a), Q(b), \ldots, \ldots$
- An *interpretation* gives truth-values for all these propositions
- you could then imagine a truth-table to define correctness / validity, and thence soundness and completeness
- A universally quantified statement $\forall x.\ P(x)$ is treated as $P(a) \wedge P(b) \wedge P(c) \wedge \ldots$
- An existentially quantified statement $\exists x.\ Q(x)$ is treated as $Q(a) \vee Q(b) \vee Q(c) \vee \ldots$

Completeness and Semi-Decidability

- First-Order Predicate Calculus is sound and complete
- So you can find a proof of any valid statement
- But the truth-tables aren't finite — you can't actually prove or disprove a statement by truth-tables
- If there is a proof you can find it by mindlessly trying all sequences of rules of length 1, length 2, ...
- But if you don't find a proof, you haven't established anything
- First-Order Predicate Calculus is *semi-decidable*

First-Order — what's that?

- Remember Haskell — functions and ordinary data are treated *alike*
- `fzip [f,g,h] [x,y,z] == [f x, g y, h z]`
  `fzip` is a higher-order function
- In First-Order Predicate Calculus,
  individuals and predicates are *different*
- $\forall x.\ P(x)$ is OK
- $\forall X.\ X(a)$ is NOT OK, in First-Order Calculus
- Prolog (later lecture) is a *first-order* logic language