

# Soundness and Completeness

**COMP2600 — Formal Methods for Software Engineering**

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Semester 2, 2011

# Introduction

Various proof methods to prove a statement in propositional calculus

- Truth tables
- the Natural Deduction rules
- Simplification by algebraic (equivalence) rules

We hope these all are correct, whatever that means.

At least we would hope these all give the same answers!

## Correctness

We *define* correctness (validity) by reference to truth tables.

Statement to be tested:  $(p \wedge (q \vee r)) \rightarrow (q \vee p)$

$p$	$q$	$r$	$q \vee r$	$p \wedge (q \vee r)$	$q \vee p$	$(p \wedge (q \vee r)) \rightarrow (q \vee p)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	F	T
F	F	F	F	F	F	T

We regard it as correct, or *valid*, because truth-tables show it is true for *all* possible truth-values of  $p, q, r$

## Validity of a rule

A similar table shows that the rule  $\frac{p \wedge (q \vee r)}{q \vee p}$  is valid, because whenever the premise is true, so is the conclusion

$p$	$q$	$r$	$q \vee r$	$p \wedge (q \vee r)$	$q \vee p$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	F
F	F	F	F	F	F

## Soundness and Completeness

We say that a proof system (such as the rules of Natural Deduction) is

**Sound**, if every **derivable** (ie, provable) statement or rule is **valid**

**Complete**, if every **valid** statement or rule is **derivable**

Example: the rule  $\frac{p \quad q}{p \wedge q}$

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Whenever both the premises are true, then so is the conclusion, so that rule is **sound**.

## Soundness of the $\vee$ -E rule

$$\frac{\begin{array}{ccc} [p] & [q] & \\ & \vdots & \vdots & \\ p \vee q & r & r \end{array}}{r}$$

There will be a sort of inductive argument here: to prove that the whole proof using this rule is valid, we assume that the sub-proofs are valid.

## Soundness of the $\vee$ -E rule

$p$	$q$	$r$	$p \vee q$	$\frac{[p]}{r}$	$\frac{[q]}{r}$	$r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Whenever all the “premises” (sub-proofs) are true (valid), then the conclusion is true, so the  $\vee$ -E rule is valid

## Soundness — summary

- We have shown that some of the Natural Deduction rules are sound
- Likewise all the other rules are sound
- So, whenever the premises (if any) of a proof are true, then each subsequent line of the proof will be true (the rules “preserve” truth)
- so the conclusion will be true
- formally, this is a doubly-inductive argument over the length of the proof
  - assume subproofs within the proof are valid
  - assume previous lines at current indentation level are true

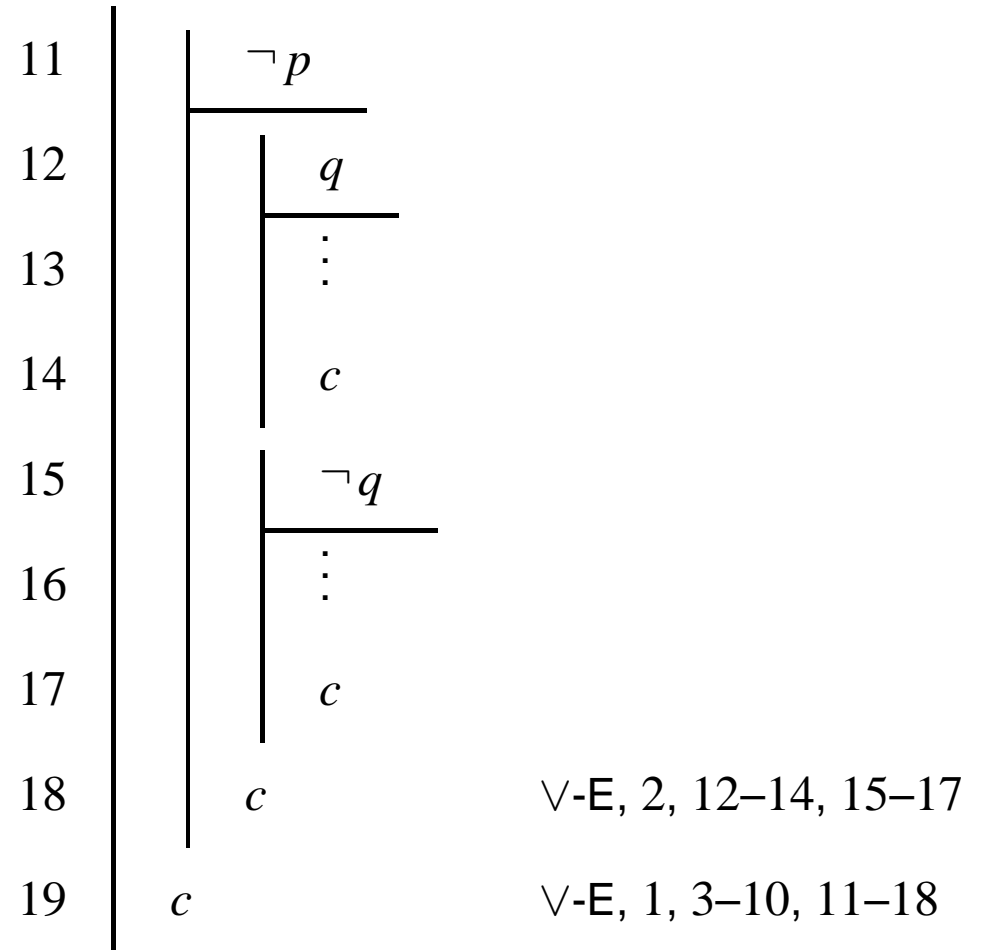
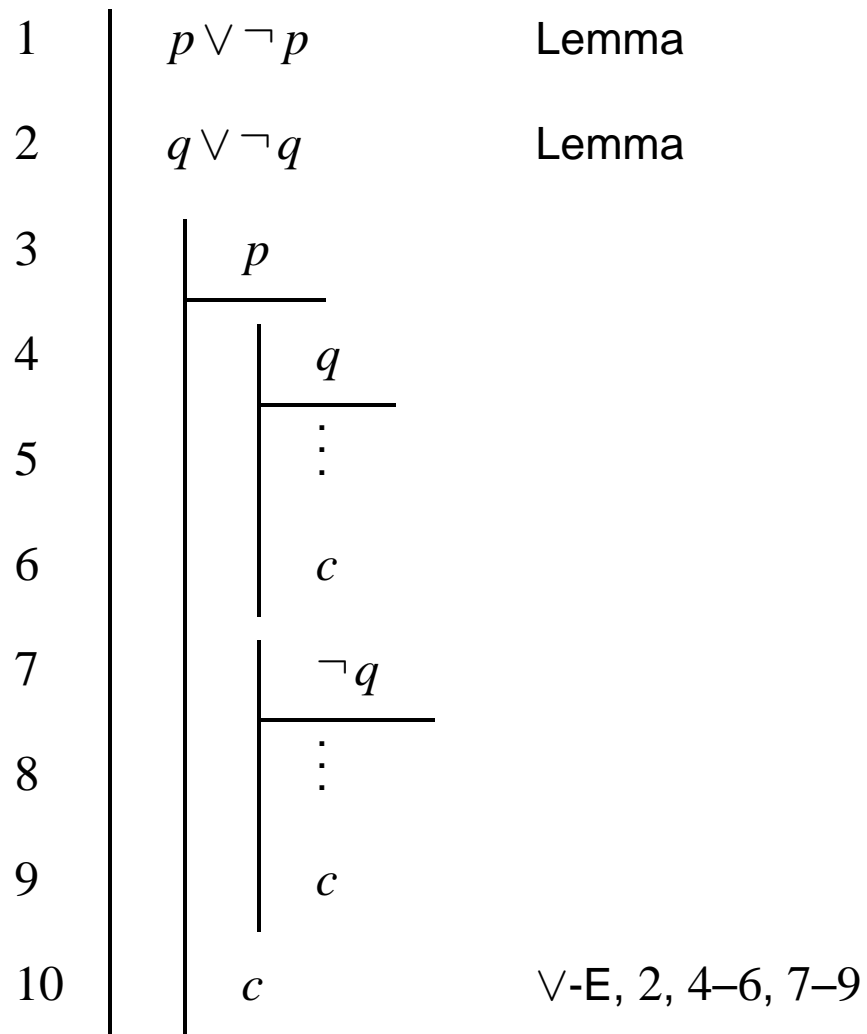
So the formula or rule produced by a natural deduction proof will be valid

## Completeness

- **Completeness** is a property of the whole set of rules, rather than of individual rules.
- **Completeness** says that any valid formula or rule can be proved by the rules of Natural Deduction

How to do this?

- Just reproduce the truth-table in a natural deduction proof
- Uses the Excluded Middle Lemma, stated in lectures,  $p \vee \neg p$
- Contains nested uses of the  $\vee$ -E rule
- It is rather long — you prove the same conclusion  $2^n$  times
- Steps of each sub-proof similar to the calculations in the truth-table



## How do do a proof which imitates the truth table calculation

The subproofs above imitate the calculations of each row of the truth table.

A row of the truth table involves several “calculations”, using the truth table rules.

We can imitate these calculations by a proof, using similar steps.

One step in calculating the truth table corresponds to one use of a rule or derived rule, such as those on the next slides.

## Completeness – proving truth-table values for $\rightarrow$ -I

$p$	$q$	$p \rightarrow q$	corresponding “rule” to prove
T	T	T	$\frac{p \quad q}{p \rightarrow q}$
T	F	F	$\frac{p \quad \neg q}{\neg(p \rightarrow q)}$
F	T	T	$\frac{\neg p \quad q}{p \rightarrow q}$
F	F	T	$\frac{\neg p \quad \neg q}{p \rightarrow q}$

## Completeness – proving truth-table values for $\wedge$ -I

$p$	$q$	$p \wedge q$	corresponding “rule” to prove
T	T	T	$\frac{p \quad q}{p \wedge q}$
T	F	F	$\frac{p \quad \neg q}{\neg(p \wedge q)}$
F	T	F	$\frac{\neg p \quad q}{\neg(p \wedge q)}$
F	F	F	$\frac{\neg p \quad \neg q}{\neg(p \wedge q)}$

## Derived Rules to imitate calculating a truth table row

You already have rules like  $\frac{p \quad q}{p \wedge q}$  and  $\frac{p}{p \vee q}$

You need rules like  $\frac{\neg p \quad q}{\neg(p \wedge q)}$  and  $\frac{\neg p \quad \neg q}{\neg(p \vee q)}$

1	$\neg p$	
2	$p \wedge q$	
3	$p$	$\wedge$ -E, 2
4	$p \wedge \neg p$	$\wedge$ -I, 3, 1
5	$\neg(p \wedge q)$	$\neg$ -I, 2-4

(premise  $q$  is not actually needed)

1	$\neg p$		
2	$\neg q$		
3	$p \vee q$		
4	$p$		
5	$p \wedge \neg p$	$\wedge$ -I, 4, 1	
6	$q$		
7	$q \wedge \neg q$	$\wedge$ -I, 6, 2	
8	$p \wedge \neg p$	Lemma	
9	$p \wedge \neg p$	$\vee$ -E, 3, 4–5, 6–8	
10	$\neg(p \vee q)$	$\neg$ I, 3–9	

The Lemma is Contradiction Elimination (lectures), that from a contradiction such as  $q \wedge \neg q$  you can prove anything, such as  $p \wedge \neg p$

## Consistency

- A logical system is *consistent* if it cannot prove falsity, ie (in our system) contradiction,  $p \wedge \neg p$ .
- In Natural Deduction (as in most logical systems), if you can prove  $p \wedge \neg p$  then you can prove anything. So what's the use of that ?
- An alternative definition of consistency is that in an inconsistent logical system you can prove *anything*
- This definition is equivalent to the first one if, in that system, you can prove anything from a contradiction

# Soundness and Completeness in Predicate Calculus

## Recall the Propositional Calculus

- For the Propositional Calculus we defined correctness / validity by truth-tables.
- A statement or rule is *valid* if shown by truth-tables; that is, you look at *all* possible (truth-)values of the propositional variables  $p, q, r, \dots$  involved
- In this way you can *determine* whether a statement or rule is valid or not (Propositional Calculus is *decidable*).
- A choice of truth-values for  $p, q, r, \dots$  (ie, a row of the truth-table) is called an *interpretation*

## Soundness and Completeness in Predicate Calculus

The Predicate Calculus is similar but more complicated.

- Imagine a set (domain)  $D$  of individuals,  $a, b, c, \dots$
- If the formulae involve predicates  $P, Q, R, \dots$ , there are propositions  $P(a), P(b), P(c), \dots, Q(a), Q(b), \dots, \dots$
- An *interpretation* gives truth-values for all these propositions
- you could then imagine a truth-table to define correctness / validity, and thence soundness and completeness
- A universally quantified statement  $\forall x. P(x)$  is treated as  $P(a) \wedge P(b) \wedge P(c) \wedge \dots$
- An existentially quantified statement  $\exists x. Q(x)$  is treated as  $Q(a) \vee Q(b) \vee Q(c) \vee \dots$

## Completeness and Semi-Decidability

- First-Order Predicate Calculus is sound and complete
- So you can find a proof of any valid statement
- But the truth-tables aren't finite — you can't actually prove or disprove a statement by truth-tables
- If there is a proof you can find it by mindlessly trying all sequences of rules of length 1, length 2, ...
- But if you don't find a proof, you haven't established anything
- First-Order Predicate Calculus is *semi-decidable*

## First-Order — what's that?

- Remember Haskell — functions and ordinary data are treated *alike*
- `fzip [f,g,h] [x,y,z] == [f x, g y, h z]`  
fzip is a higher-order function
- In First-Order Predicate Calculus,  
individuals and predicates are *different*
- $\forall x. P(x)$  is OK
- $\forall X. X(a)$  is NOT OK, in First-Order Calculus
- Prolog (later lecture) is a *first-order* logic language