Natural Deduction
(Predicate Calculus)

COMP2600 — Formal Methods for Software Engineering

Jeremy Dawson

Australian National University

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The story so far:

$$\dfrac{\begin{array}{c}[p]\\ \vdots\\ q\end{array}}{p \longrightarrow q} \qquad \dfrac{p \qquad p \longrightarrow q}{q} \qquad \dfrac{\begin{array}{c}[p]\\ \vdots\\ q \wedge \neg q\end{array}}{\neg p} \qquad \dfrac{\begin{array}{c}[\neg p]\\ \vdots\\ q \wedge \neg q\end{array}}{p}$$

$$\dfrac{p \qquad q}{p \wedge q} \qquad \dfrac{p \wedge q}{p} \qquad \dfrac{p \wedge q}{q}$$

$$\dfrac{p}{p \vee q} \qquad \dfrac{p}{q \vee p} \qquad \dfrac{p \vee q \qquad \begin{array}{c}[p]\\ \vdots\\ r\end{array} \qquad \begin{array}{c}[q]\\ \vdots\\ r\end{array}}{r}$$

Limitations of propositional logic - reminder

Consider this “argument”:

Natural language	Propositional logic
All COMP2600 students are happy.	p
Lisa is a COMP2600 student.	q
Therefore, Lisa is happy.	$\therefore r$

Useful? No! This is not a valid argument form in terms of propositional logic, since $p \wedge q \rightarrow r$ is not a tautology. (There is no relationship between the propositions.)

Moving on - Predicate Calculus

- Truth tables are virtually useless here.
- The exception is where domains are small.
- Natural deduction is inevitably required.

  There are introduction and elimination rules for quantifiers.
- There are some handy equational rules which help.

Quantifying over finite domains

$$\frac{D = \{x_1, x_2, \ldots, x_n\}}{(\forall x \in D.\ P(x)) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)}$$

$$\frac{D = \{x_1, x_2, \ldots, x_n\}}{(\exists x \in D.\ P(x)) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)}$$

An example of domain finiteness

"There is an integer between 4 and 7 that is divisible by all integers between 1 and 3."

$$\exists n \in \{4,5,6,7\}.\ \forall m \in \{1,2,3\}.\ n \text{ mod } m = 0$$

$$\begin{aligned}
&((4 \text{ mod } 1 = 0) \wedge (4 \text{ mod } 2 = 0) \wedge (4 \text{ mod } 3 = 0)) \ \vee \\
&((5 \text{ mod } 1 = 0) \wedge (5 \text{ mod } 2 = 0) \wedge (5 \text{ mod } 3 = 0)) \ \vee \\
&((6 \text{ mod } 1 = 0) \wedge (6 \text{ mod } 2 = 0) \wedge (6 \text{ mod } 3 = 0)) \ \vee \\
&((7 \text{ mod } 1 = 0) \wedge (7 \text{ mod } 2 = 0) \wedge (7 \text{ mod } 3 = 0))
\end{aligned}$$

Negating “there exists”

$$\neg(\exists x.\ P(x)) \leftrightarrow (\forall x.\ \neg P(x))$$

- Examples:
  - “No students dislike COMP2600”.
  - $\neg\exists s \in$ Student. s dislikes COMP2600
  - $\forall s \in$ Student. $\neg s$ dislikes COMP2600
  - “All students like COMP2600”.
- Think about finite domains and compare to De Morgan’s laws!
  $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q) \quad ; \quad \neg(p_1 \vee \ldots \vee p_n) \leftrightarrow (\neg p_1 \wedge \ldots \wedge \neg p_n)$

Negating “for all”

$$\neg(\forall x.\ P(x)) \leftrightarrow (\exists x.\ \neg P(x))$$

- Examples:
  - “Not all students passed the exam”.
  - $\neg \forall s \in$ Student. s passed the exam
  - $\exists s \in$ Student. $\neg s$ passed the exam
  - “There are some students who did not pass the exam”.
- Compare to De Morgan’s laws!

  $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q) \quad ; \quad \neg(p_1 \wedge \ldots \wedge p_n) \leftrightarrow (\neg p_1 \vee \ldots \vee \neg p_n)$

Mixed negated quantifiers

Here are four different expressions of the fact that there is no upper bound to the natural numbers.

We can shift from one to the other by negating the quantifiers.

$$\neg\exists m.\ \forall n.\ m \geq n$$

$$\forall m.\ \neg\forall n.\ m \geq n$$

$$\forall m.\ \exists n.\ \neg m \geq n$$

$$\forall m.\ \exists n.\ m < n$$

Existential quantifiers with empty domains

Consider the existential formula:

$$\exists x \in \emptyset.\ P(x)$$

The definition of the existential quantifier says that this should be false, regardless of P, since there are no elements that could exist.

As a result, the following is true:

$$(\exists x \in \emptyset.\ P(x)) \equiv \mathbf{F}$$

Universal quantifiers with empty domains

Consider the universally quantified formula:

$$\forall x \in \emptyset.\ \ P(x)$$

What does “for all x in the empty set” mean?

Use the negation rule

$$(\forall x \in \emptyset.\ \ P(x)) \ \equiv\ \neg(\exists x \in \emptyset.\ \ \neg P(x))$$

We have

$$(\forall x \in \emptyset.\ \ P(x)) \ \equiv\ \mathbf{T}$$

Universal quantifiers with empty domains

“All unicorns climb trees”.

Since the set of unicorns is empty, the statement is true.

The statement can never be disproved by finding a counterexample.

Natural deduction in Predicate Calculus

There are Gentzen rules for Predicate Calculus

- $\forall$-E is also called *universal instantiation*;
- $\forall$-I is also called *universal generalisation*;
- $\exists$-E is also called *existential instantiation*;
- $\exists$-I is also called *existential generalisation*;

Proof in Predicate Calculus is usually based on these rules, together with the rules for propositional calculus.

Universial Quantification – Instantiation

∀-E (universal instantiation)

$$\frac{\forall x.\ P(x)}{P(a)}$$

$$\begin{array}{l|ll}
n & \forall x.\ \mathrm{Fish}(\mathrm{x}) \longrightarrow \mathrm{HasFins}(\mathrm{x}) & \\
\vdots & \vdots & \\
m & \mathrm{Fish}(\mathrm{a}) \longrightarrow \mathrm{HasFins}(\mathrm{a}) & \forall\text{-E, } n
\end{array}$$

If a predicate is true for all members of a domain,

then it is also true for a specific one (a must be a member of the domain)

Universal Quantification – Generalisation

∀-I (universal generalisation) $$\frac{P(a) \qquad (a \text{ arbitrary, a variable})}{\forall x.\, P(x)}$$

$$\begin{array}{l|l|l}
n & a & \vdots \\
\vdots & & \vdots \\
m & & \mathrm{Cat(a)} \longrightarrow \mathrm{EatsFish(a)} \\
m+1 & \multicolumn{2}{l}{\forall x.\ \mathrm{Cat(x)} \longrightarrow \mathrm{EatsFish(x)}}
\end{array}$$

The a on the left of the bar is a *guard* which reminds us that this variable is local to the inner derivation, and it cannot be free in an assumption.

It is like an “assumption” that a is an arbitrary member of the domain.

That is, the proof from lines n to m must work for *anything* in place of a.

Breaching the arbitrariness requirement

When we generalise for a variable a, the same proof steps must be possible for all members of the domain.

1	$(\mathrm{Cat}(\mathrm{kitty}) \rightarrow \mathrm{HasFur}(\mathrm{kitty})) \ \wedge \ \mathrm{Cat}(\mathrm{kitty})$	
2	$\mathrm{Cat}(\mathrm{kitty}) \rightarrow \mathrm{HasFur}(\mathrm{kitty})$	$\wedge$-E, 1
3	$\mathrm{Cat}(\mathrm{kitty})$	$\wedge$-E, 1
4	$\mathrm{HasFur}(\mathrm{kitty})$	$\rightarrow$-E, 2, 3
5	$\forall x.\ \mathrm{HasFur}(\mathrm{x})$ *WRONG*	$\wedge$-I, 4

WRONG because kitty appears in an assumption (step 1) (and step 4 is still in the scope of that assumption)

Free and bound variables

Bound: Every occurrence of variable x in $\forall x.\ p(x)$ and in $\exists x.\ p(x)$ is *bound*.

Free: Every occurrence of a variable that is not bound is *free*.

Example:

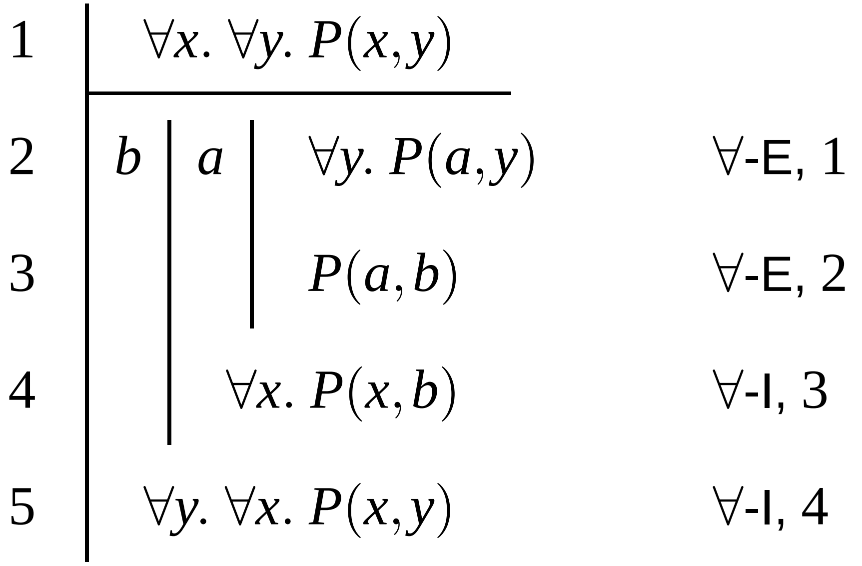
$$(\forall x.\ \ x+y=y+x) \quad \wedge \quad (\forall x.\ \ \exists y.\ \ y>x+z)$$

Q: Which occurrences of variables are free and which are bound?

A: All occurrences of x are bound; none of z are; and just the last 2 occurrences of y are bound.

Hence the instance of z is free, as are the first two occurrences of y.

Example: $(\forall x.\ \forall y.\ P(x,y)) \leftrightarrow (\forall y.\ \forall x.\ P(x,y))$



1	$\forall x.\ \forall y.\ P(x,y)$	
2	b | a | $\forall y.\ P(a,y)$	$\forall$-E, 1
3	$P(a,b)$	$\forall$-E, 2
4	$\forall x.\ P(x,b)$	$\forall$-I, 3
5	$\forall y.\ \forall x.\ P(x,y)$	$\forall$-I, 4

Exercise: also show the reverse to get equivalence, $\leftrightarrow$

Existential Quantification – Generalisation

∃-I (existential generalisation)

$$\frac{P(a)}{\exists x.\ P(x)}$$

$$\begin{array}{l|ll} n & \mathrm{Dog}(\text{fido}) & \\ \vdots & \vdots & \\ m & \exists x.\ \mathrm{Dog}(x) & \exists\text{-I},\ n \end{array}$$

An Invalid Argument

This argument is invalid if the domain is empty.

$$\begin{array}{r|ll} 1 & \forall x.\, P(x) & \\ 2 & P(a) & \forall\text{-E, } 1 \\ 3 & \exists x.\, P(x) & \exists\text{-I, } 2 \end{array}$$

Which step is invalid ??

Existential Quantification – Instantiation

$$\frac{\exists x.\, P(x) \qquad \begin{array}{c}[P(a)]\\ \vdots \\ q\end{array} \qquad (a \text{ arbitrary, a variable})}{q \quad (a \text{ not free in } q)}$$

Rationale: if $P(x)$ holds for some individual x,

- *let* that individual be called a (so $P(a)$ holds)
- *prove* that q follows
- as q doesn't involve our choice of a,
  q holds regardless of which individual has P true

The proof of q from $P(a)$ must work for *any* individual in place of a

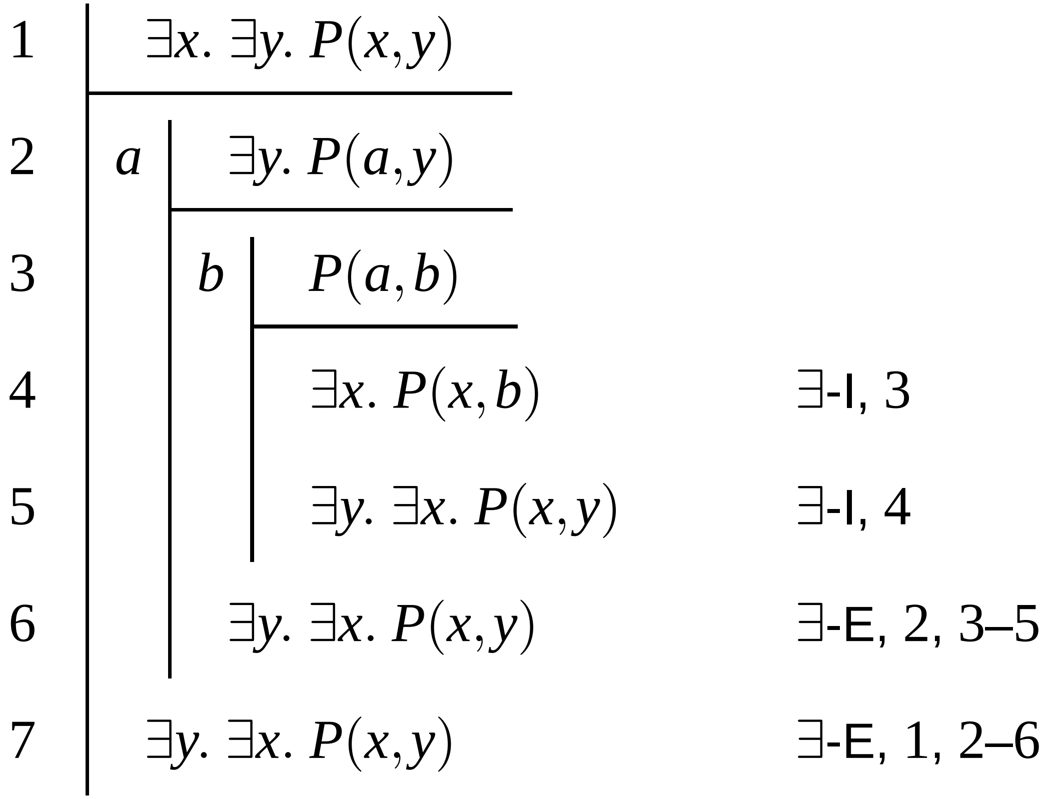
Existential Quantification – Instantiation

Prove $$\frac{\exists x.\ \mathrm{Elephant}(x) \qquad \forall x.\ \mathrm{Elephant}(x) \rightarrow \mathrm{Huge}(x)}{\exists x.\ \mathrm{Huge}(x)}$$

1	$\exists x.\ \mathrm{Elephant}(x)$		
2	$\forall x.\ \mathrm{Elephant}(x) \rightarrow \mathrm{Huge}(x)$		
3	a	$\mathrm{Elephant}(a)$	
4		$\mathrm{Elephant}(a) \rightarrow \mathrm{Huge}(a)$	∀-E, 2
5		$\mathrm{Huge}(a)$	→-E, 3, 4
6		$\exists x.\ \mathrm{Huge}(x)$	∃-I, 5
7	$\exists x.\ \mathrm{Huge}(x)$		∃-E, 1, 3–6

The notation reflects an assumption: since there is some individual x such that $\mathrm{Elephant}(x)$, *assume* that individual is a

Example: $(\exists x.\ \exists y.\ P(x,y)) \leftrightarrow (\exists y.\ \exists x.\ P(x,y))$



1			$\exists x.\ \exists y.\ P(x,y)$	
2	a		$\exists y.\ P(a,y)$	
3		b	$P(a,b)$	
4			$\exists x.\ P(x,b)$	∃-I, 3
5			$\exists y.\ \exists x.\ P(x,y)$	∃-I, 4
6			$\exists y.\ \exists x.\ P(x,y)$	∃-E, 2, 3–5
7			$\exists y.\ \exists x.\ P(x,y)$	∃-E, 1, 2–6

Exercise: also show the reverse to get equivalence, ↔

Example: prove

$$\frac{\exists x.\ \forall y.\ P(x,y)}{\forall y.\ \exists x.\ P(x,y)}$$

1	$\exists x.\ \forall y.\ P(x,y)$			
2	b	a	$\forall y.\ P(a,y)$	
3			$P(a,b)$	∀-E, 2
4			$\exists x.\ P(x,b)$	∃-I, 3
5		$\exists x.\ P(x,b)$		∃-E, 1, 2–4
6	$\forall y.\ \exists x.\ P(x,y)$			∀-I, 5

Example: another proof of $\dfrac{\exists x.\ \forall y.\ P(x,y)}{\forall y.\ \exists x.\ P(x,y)}$

We got the previous proof by first looking at the goal, ($\forall y.\ \ldots$), so using $\forall$-I.

Here we first look at what we have, ($\exists x.\ \ldots$), and so use $\exists$-E.

1			$\exists x.\ \forall y.\ P(x,y)$	
2	a		$\forall y.\ P(a,y)$	
3		b	$P(a,b)$	$\forall$-E, 2
4			$\exists x.\ P(x,b)$	$\exists$-I, 3
5			$\forall y.\ \exists x.\ P(x,y)$	$\forall$-I, 4
6			$\forall y.\ \exists x.\ P(x,y)$	$\exists$-E, 1, 2–5

$\exists x.\ \forall y.\ \mathrm{Eats}(x,y) \quad \to \quad \forall y.\ \exists x.\ \mathrm{Eats}(x,y)$

There is an animal that can eat all foods.	All foods can be eaten by some animal.

$\forall y.\ \exists x.\ \mathrm{Eats}(x,y) \quad \to \quad \exists x.\ \forall y.\ \mathrm{Eats}(x,y)$

All foods can be eaten by some animal.	There is an animal that can eat all foods.

(Photo: Steve Elliott CCA2.0)

Is this second version true? Try to prove it. What happens?

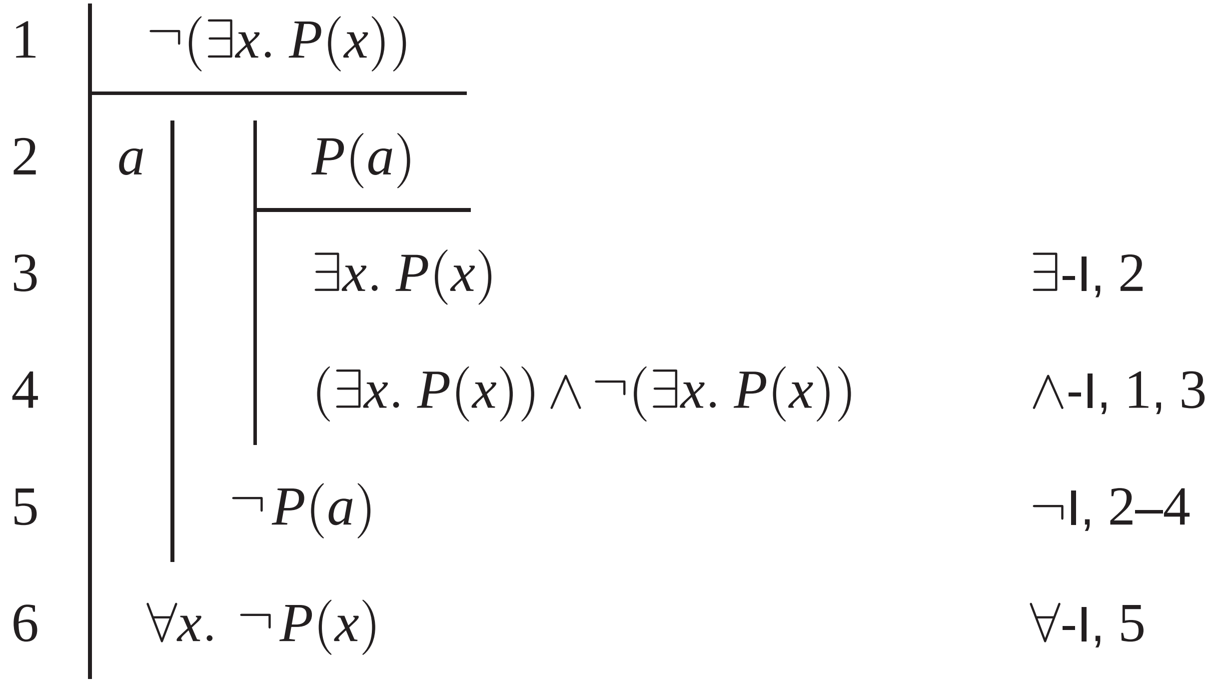
Of $(\forall x.\ \exists y.\ y > x)$ and $(\exists y.\ \forall x.\ y > x)$ which is true?

So, is $(\forall x.\ \exists y.\ y > x) \to (\exists y.\ \forall x.\ y > x)$ true?

The “Quantifier Negation” Equivalence

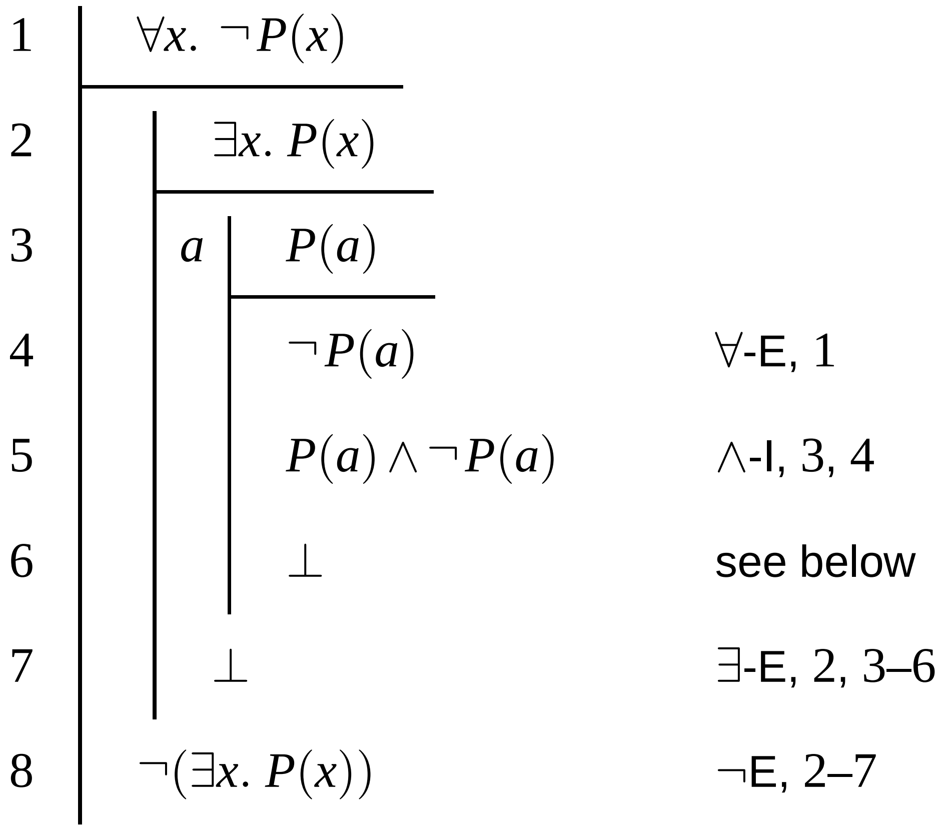
$$(\forall x.\ \neg P(x)) \leftrightarrow \neg(\exists x.\ P(x))$$

Prove $\dfrac{\neg(\exists x.\ P(x))}{\forall x.\ \neg P(x)}$



1		$\neg(\exists x.\ P(x))$	
2	a	$P(a)$	
3		$\exists x.\ P(x)$	∃-I, 2
4		$(\exists x.\ P(x)) \land \neg(\exists x.\ P(x))$	∧-I, 1, 3
5		$\neg P(a)$	¬I, 2–4
6		$\forall x.\ \neg P(x)$	∀-I, 5

Conversely, prove $\dfrac{\forall x.\ \neg P(x)}{\neg(\exists x.\ P(x))}$



1		$\forall x.\ \neg P(x)$	
2		$\exists x.\ P(x)$	
3	a	$P(a)$	
4		$\neg P(a)$	$\forall$-E, 1
5		$P(a) \wedge \neg P(a)$	$\wedge$-I, 3, 4
6		$\perp$	see below
7		$\perp$	$\exists$-E, 2, 3–6
8		$\neg(\exists x.\ P(x))$	$\neg$E, 2–7

Think of $\perp$ as an abbreviation for *any* contradiction, say, $q \wedge \neg q$.

From step 5 to 6 uses the proof of the derived rule “contradiction elimination”.