# Natural Deduction
# (Propositional Calculus)

**COMP2600 — Formal Methods for Software Engineering**

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# Contrasting Major Proof Techniques

There are three major styles of proof in logic and mathematics.

- Model based computation:

  For propositional logic, this is the use of truth tables.

- Algebraic proof:

  Relies on simplification rules.

- Deductive reasoning:

  The application of rules of inference.

## The logical operators

- $p \wedge q$ : "*p and q*" : true iff both $p$ and $q$ are true
- $p \vee q$ : "*p or q*" : true iff either $p$ or $q$ (or both) is true
- $p \rightarrow q$ : "*p implies q*", or "*if p then q*" :
  true iff either $p$ is false or $q$ is true (or both of these)
- $\neg p$ : "*not p*" : true iff $p$ is false
- $p \leftrightarrow q$ : "*p is equivalent to q*", true when $p$ and $q$ are either both false or both true (actually it is just $p = q$, as boolean values)

## The logical operators defined by truth tables

We can express the logical operators using truth tables

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $p \rightarrow q$ | $\neg p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T |
| T | F | T | F | F | F | F |
| F | T | T | F | T | T | F |
| F | F | F | F | T | T | T |

We can use these tables to calculate the truth values of logical expressions

## Model based computation

Statement to be proved: $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee r))$

For all 8 (= $2^3$) possibilities of $p, q, r$, calculate truth value of the statement

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $(p \wedge q) \vee r)$ | $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee r))$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T |
| T | F | T | T | T | F | T | T |
| T | F | F | F | F | F | F | T |
| F | T | T | T | F | F | T | T |
| F | T | F | T | F | F | F | T |
| F | F | T | T | F | F | T | T |
| F | F | F | F | F | F | F | T |

## Testing whether proposition valid or not

Likewise we can test whether the proposition $((p \to q) \land (q \to r)) \leftrightarrow \neg(r \to p)$ is *valid*, ie, true for *all* truth-values of $p, q, r$

| $p$ | $q$ | $r$ | $r \to p$ | $\neg(r \to p)$ | $q \to r$ | $p \to q$ | $(p \to q) \land (q \to r)$ | result |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | F | T | T | T | F |
| T | T | F | T | F | F | T | F | T |
| T | F | T | T | F | T | F | F | T |
| T | F | F | T | F | T | F | F | T |
| F | T | T | F | T | T | T | T | T |
| F | T | F | T | F | F | T | F | T |
| F | F | T | F | T | T | T | T | T |
| F | F | F | T | F | T | T | T | F |

So it is not true for the cases (*"counterexamples"*) $p, q, r$ all $True$, or all $False$

# Algebraic Proof

Statement to be proved: $(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p)$

Calculate that it is equal to $T$ (true) using various logical rules

$$
\begin{array}{lll}
(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p) & & \\
\quad \equiv \neg (p \wedge (q \vee r)) \vee ((q \rightarrow s) \vee p) & & a \rightarrow b \equiv \neg a \vee b \\
\quad \equiv (\neg p \vee \neg (q \vee r)) \vee ((q \rightarrow s) \vee p) & & \neg (a \wedge b) \equiv \neg a \vee \neg b \\
\quad \ldots\ldots\ldots & & \\
\quad \equiv (p \vee \neg p) \vee (\neg (q \vee r) \vee (q \rightarrow s)) & & \text{assoc \& commut} \\
\quad \equiv T \vee (\neg (q \vee r) \vee (q \rightarrow s)) & & a \vee \neg a \equiv T \\
\quad \equiv T & & T \vee a \equiv T
\end{array}
$$

We will not be using this technique in this course

## The notion of a Deductive Reasoning proof

- A proof is a sequence of *steps*.
- Each step is a *logical sentence*.
- Each step is either:
  - an *axiom* or an *assumption*; or
  - a statement which follows from previous steps via a valid *rule of inference*.

## Deductive Reasoning

Statement to be proved: $(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p)$

| | | |
|---|---|---|
| 1 | $p \wedge (q \vee r)$ | Assumption |
| 2 | $p$ | $\wedge$-E, 1 |
| 3 | $(q \rightarrow s) \vee p$ | $\vee$-I, 2 |
| 4 | $(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p)$ | $\rightarrow$-I, 1–3 |

## Which rules are axioms?

- There are many valid rules of inference.
- Many sets of inference rules are used as bases for proof.
- Gentzen's set of rules are well-known and people associate them with *natural deduction.*
- For each connective, there is an introduction and an elimination rule.

# Conjunction – Rules

$\wedge$-I (and introduction)

$$\frac{p \qquad q}{p \wedge q}$$

$\wedge$-E (and elimination)

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q}$$

## Commutativity of Conjunction (derived rule)

$$\frac{p \land q}{q \land p}$$

| | | |
|---|---|---|
| 1 | $p \land q$ | |
| 2 | $p$ | $\land$-E, 1 |
| 3 | $q$ | $\land$-E, 1 |
| 4 | $q \land p$ | $\land$-I, 2, 3 |

## Implication – Rules

$\longrightarrow$-I (implies introduction)

$$\frac{\begin{array}{c}[\,p\,]\\ \vdots \\ q\end{array}}{p \longrightarrow q}$$

This notation means that by *assuming* $p$, you can *prove* $q$

$\longrightarrow$-E (implies elimination)

$$\frac{p \qquad p \longrightarrow q}{q}$$

# Transitivity of implication (derived rule)

To prove

$$\frac{p \rightarrow q \qquad q \rightarrow r}{p \rightarrow r}$$

| | | |
|---|---|---|
| 1 | $p \rightarrow q$ | |
| 2 | $q \rightarrow r$ | |
| 3 | $\quad p$ | |
| 4 | $\quad q$ | $\rightarrow$-E, 1, 3 |
| 5 | $\quad r$ | $\rightarrow$-E, 2, 4 |
| 6 | $p \rightarrow r$ | $\rightarrow$-I, 3–5 |

Lines 1 and 2 are the assumptions given,
you may use these throughout the proof

Line 3 is an assumption made within the proof,
you may use it only within its scope (lines 3 to 5)

## Notation: justification of a step – IMPORTANT

| | | |
|---|---|---|
| 1 | $p \to q$ | |
| 2 | $p$ | |
| 3 | $q$ | $\to$-E, 1, 2 |
| 4 | $p \to q$ | $\to$-I, 2–3 |

This is a rather silly proof, we succeed in proving what we started with.

But it illustrates the meaning of the line number notation: (IMPORTANT!)

- $\to$-E,1,2 means that rule $\to$-E proves line 3 from lines 1 and 2
- $\to$-I,2-3 means rule $\to$-I proves line 4 from the fact that we could assume line 2 and (using that assumption) prove line 3

## Rules involving assumptions – IMPORTANT

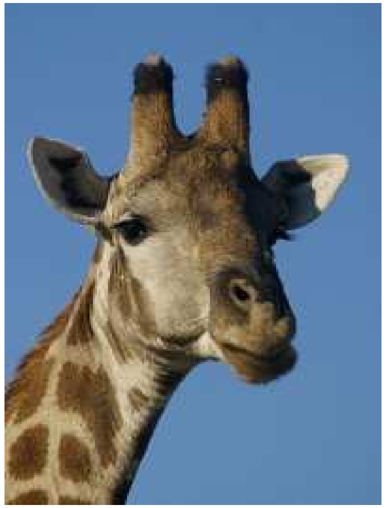
- If a statement is inside the scope of an assumption, then it depends on that assumption.

| | | |
|---|---|---|
| 1 | $p \rightarrow q$ | |
| 2 | $q \rightarrow r$ | |
| 3 | $\quad p$ | |
| 4 | $\quad q$ | $\rightarrow$-E, 1, 3 |
| 5 | $\quad r$ | $\rightarrow$-E, 2, 4 |
| 6 | $q \wedge r$ *WRONG* | $\wedge$-I, 4, 5 |

Given $p \rightarrow q$ and $q \rightarrow r$, I then assumed $p$ and proved $q \wedge r$, but $q \wedge r$ depends on $p$.

- You can assume anything, but it might not be useful.

| | | |
|---|---|---|
| 1 | $p \wedge q$ | |
| 2 | $q$ | $\wedge$-E, 1 |
| 3 | $(p \wedge q) \rightarrow q$ | $\rightarrow$-I, 1–2 |



(Photo: Hans Hillewaert CCSA2.5)

| | | |
|---|---|---|
| 1 | $p$ $\wedge$ You are a giraffe | |
| 2 | You are a giraffe | $\wedge$-E, 1 |
| 3 | $p$ $\wedge$ You are a giraffe $\rightarrow$ You are a giraffe | $\rightarrow$-I, 1–2 |

## Disjunction – Rules

$\vee$-I (or introduction)

$$\frac{p}{p \vee q} \qquad \frac{p}{q \vee p}$$

$\vee$-E (or elimination)

$$\frac{p \vee q \quad \begin{matrix}[p] \\ \vdots \\ r\end{matrix} \quad \begin{matrix}[q] \\ \vdots \\ r\end{matrix}}{r}$$

## $\vee$-E template (it's a bit tricky)

| | | | |
|---|---|---|---|
| 1 | $p \vee q$ | | |
| 2 | | $p$ | |
| $\vdots$ | | $\vdots$ | |
| $a$ | | $r$ | |
| $b$ | | $q$ | |
| $\vdots$ | | $\vdots$ | |
| $c$ | | $r$ | |
| $d$ | $r$ | | $\vee$-E, 1, 2–$a$, $b$–$c$ |

# Commutativity of Disjunction (derived rule)

$$\frac{p \vee q}{q \vee p}$$

| | | |
|---|---|---|
| 1 | $p \vee q$ | |
| 2 | $\quad p$ | |
| 3 | $\quad q \vee p$ | $\vee$-I, 2 |
| 4 | $\quad q$ | |
| 5 | $\quad q \vee p$ | $\vee$-I, 4 |
| 6 | $q \vee p$ | $\vee$-E, 1, 2–3, 4–5 |

## Negation – Rules

Idea: assume the opposite of what you want to prove and find a contradiction — so your assumption must have been wrong

¬-I (not introduction)

$$\frac{\begin{array}{c}[p] \\ \vdots \\ q \wedge \neg q\end{array}}{\neg p}$$

¬-E (not elimination)
(eliminates ¬ from an *assumption*)

$$\frac{\begin{array}{c}[\neg p] \\ \vdots \\ q \wedge \neg q\end{array}}{p}$$

## Double negation introduction (derived rule)

$$\frac{p}{\neg\neg p}$$

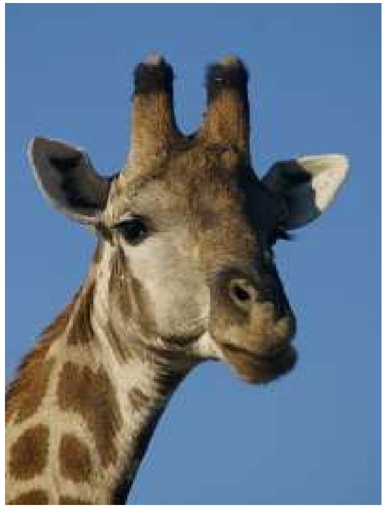
$$\frac{\text{It is raining}}{\text{It is not the case that is is not raining}}$$

| | | |
|---|---|---|
| 1 | $p$ (assumption) | |
| 2 | $\quad \neg p$ (assumption) | |
| 3 | $\quad p \wedge \neg p$ | $\wedge$-I, 1, 2 |
| 4 | $\neg\neg p$ | $\neg$I, 2–3 |

# Contradiction elimination (derived rule)

$$\frac{p \wedge \neg p}{q}$$

| | | |
|---|---|---|
| 1 | $p \wedge \neg p$ | |
| 2 | $\quad \neg$(You are a giraffe) | |
| 3 | $\quad p \wedge \neg p$ | R, 1 |
| 4 | You are a giraffe | $\neg$E, 2–3 |



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## Double negation elimination (derived rule)

$$\frac{\neg\neg p}{p}$$

| | | |
|---|---|---|
| 1 | $\neg\neg p$ | |
| 2 | $\quad \neg p$ | |
| 3 | $\quad \neg p \wedge \neg\neg p$ | $\wedge$-I, 1, 2 |
| 4 | $p$ | $\neg$E, 2–3 |

## Equivalence

$p \leftrightarrow q$ means $p$ is true if and only if $q$ is true

We can make the definition

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

which would naturally give us these rules

introduction rule:

$$\frac{p \rightarrow q \qquad q \rightarrow p}{p \leftrightarrow q}$$

elimination rules:

$$\frac{p \leftrightarrow q}{p \rightarrow q} \qquad \frac{p \leftrightarrow q}{q \rightarrow p}$$

# Equivalence — Rules

Alternatively we can get rules which don't involve the $\longrightarrow$ symbol

$\longleftrightarrow$-I ($\longleftrightarrow$ introduction)

$$\frac{\begin{matrix} [p] & [q] \\ \vdots & \vdots \\ q & p \end{matrix}}{p \leftrightarrow q}$$

$\longleftrightarrow$-E ($\longleftrightarrow$ elimination)

$$\frac{p \leftrightarrow q \qquad p}{q} \qquad\qquad \frac{p \leftrightarrow q \qquad q}{p}$$

Note the similarities to the $\longrightarrow$-I and $\longrightarrow$-E rules

## Which rule to use next?

- Guided by the "form" of your goal, and what you already have proved
- "form" — ie, look at the connective: $\wedge, \vee, \rightarrow, \neg$
- always can consider using $\neg$-E (not elimination) rule
- to prove $p \vee q$, $\vee$-I (or introduction) may not work

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q}$$

$p$ may not be necessarily true, $q$ may not be necessarily true

## To prove $p \vee q$, sometimes you need to do this:

- First step, using $\neg$-E, assume $\neg(p \vee q)$
  (hoping to prove some contradiction)
- When is $\neg(p \vee q)$ true ? When both $p$ and $q$ false!
- From $\neg(p \vee q)$ how to prove $\neg p$ ? (next slide)
- Having proved both $\neg p$ and $\neg q$, prove some further contradiction
- Exercise: prove $\dfrac{\neg p \to q}{p \vee q}$

## Not-or elimination (derived rule)

$$\frac{\neg(p \vee q)}{\neg p}$$

| | | |
|---|---|---|
| 1 | $\neg(p \vee q)$ | |
| 2 | $\quad p$ | |
| 3 | $\quad p \vee q$ | $\vee$-I, 2 |
| 4 | $\quad (p \vee q) \wedge \neg(p \vee q)$ | $\wedge$-I, 1, 3 |
| 5 | $\neg p$ | $\neg$I, 2–4 |

## Proving a contrapositive rule

In the same way, whenever you can prove any $\dfrac{p}{q}$

then you can prove $\dfrac{\neg q}{\neg p}$

| | | |
|---|---|---|
| 1 | $\neg q$ | |
| 2 | $\quad p$ | |
| 3 | $\quad q$ | your proof of $q$ from $p$ |
| 4 | $\quad q \wedge \neg q$ | $\wedge$-I, 1, 3 |
| 5 | $\neg p$ | $\neg$I, 2–4 |

## Law of the excluded middle (derived)

$$p \lor \neg p$$

*“Everything must either be or not be.”* – Russell

| | | |
|---|---|---|
| 1 | $\quad \mid \neg(p \lor \neg p)$ | |
| 2 | $\quad \mid \neg p$ | $\neg\lor$-E (previous slide), 1 |
| 3 | $\quad \mid \neg\neg p$ | $\neg\lor$-E (previous slide), 1 |
| 4 | $\quad \mid \neg p \land \neg\neg p$ | $\land$-I, 2, 3 |
| 5 | $p \lor \neg p$ | $\neg$E, 1–4 |