Reviewing Set Theory

COMP2600 — Formal Methods for Software Engineering

Jeremy Dawson

Australian National University
Semester 2, 2011

Why should we study set theory?

Understanding mathematics: It is the foundation!

Programming: Types and classes are sets.

Specifying software: Legal inputs and appropriate outputs are sets.

Reasoning about programs: This inevitably relies on sets of data values.

Semantics: The meaning of a program is a mapping of one set of data to another.

Set Theory Basics

Basic concepts:

- Objects, elements, sets
- Enumeration notation: as in $\{1,2,3\}$
- Is-element-of notation: as in $5 \in \{1,3,5,7\}$

Set equality and notational ambiguity:

- $\{1,2,3\} = \{3,2,1\}$
- $\{1,2,3,2,1\} = \{1,2,3\}$

Standard sets

Named sets

- numbers:
  $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of integers;
  $\mathbb{Q}$ is the set of rational numbers; $\mathbb{R}$ is the set of real numbers.
- booleans: The set $\{\boldsymbol{T},\boldsymbol{F}\}$ is typically called ***Bool***.
- characters: Often consisting of the set of ASCII characters, and often will be referred to as ***Char***.

The empty set $\varnothing$

- Basic properties: $\forall x.\ x \notin \varnothing$ and $\forall A.\ \varnothing \subseteq A$
- Note: $\{\varnothing\}$ is not the same set as $\varnothing$.

Notation for Infinite Sets

The *“dot-dot-dot”* notation

- $\{1,3,5,7,\ldots\}$ ‘obviously’ denotes the set of odd numbers.

Problems:

- Is it adequate?
  Occasionally, but in general, no.
- Is it ambiguous?
  Is $\{1,3,\ldots\}$ the same set as $\{1,3,4,\ldots\}$?

Characteristic Predicates

Need a better notation:

- $S = \{x \in D \mid P(x)\}$
- P will be called the characteristic predicate of the set S.

We often write $\{x \mid P(x)\}$ when the domain is understood.

Given a set S, we can get the characteristic predicate P from S: we *define* $P(x)$ to be true iff $x \in S$

We should ask ourselves “Is this always a sensible notation?”

Russell’s Paradox

Reveals a problem with an unconstrained notion of sets: even if P is a sensible predicate, it doesn’t necessarily characterise a set.

Russell considered the set of all sets that are not members of themselves:

$$S \equiv \{A \mid A \notin A\}$$

and posed the question “Is S a member of itself?”

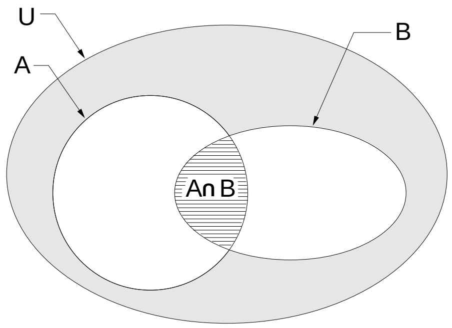
Consider the 2 cases:

- Suppose $S \in S$. Then S is in the set $\{A \mid A \notin A\}$, so $S \notin S$.
- Suppose $S \notin S$. Then S does not satisfy the predicate $A \notin A$, so $S \in S$.

Hence there is no such S. The predicate is sensible, but it does not characterise a set.

Universal Sets

Venn diagrams are a notation where union and intersection can be well illustrated. If complements are to have any meaning then Venn diagrams often show a box that corresponds to the whole domain of discourse — the universal set.



New sets from Old

Subset:

- Notation: $A \subseteq B$ holds iff $\forall x \in A.\, x \in B$
- If P is a predicate of appropriate type, and A is given, then $\{x \in A \mid P(x)\}$ is a new set.
  Of course it is a subset of A.

On subsequent slides we recall other ways to build sets.

- Power set
- Cartesian Product
- Union and intersection

New sets from Old - II

Power set:

- Notation: $\mathcal{P}(A)$ denotes the power set of A.
- Definition: $\mathcal{P}(A) \equiv \{s \mid s \subseteq A\}$
  i.e. $\mathcal{P}(A)$ is the set of all subsets of A.
- Example: $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$

Cardinality notation: $|A|$ indicates the cardinality of A.

The power set is so called because $|\mathcal{P}(A)| = 2^{|A|}$

Cartesian Product

Notation:

- $A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$
- $A \times B \times C = \{(x,y,z) \mid x \in A \wedge y \in B \wedge z \in C\}$
- etc.

Apparent abuse of notation: The product is defined in terms of a set of ordered pairs. Since the set they belong to is the Cartesian product, the definition seems circular.

Resolution: The existence of the Cartesian product is an axiom of set theory. Consequently the expression $\{(x,y) \in A \times B \mid x \in A \wedge y \in B\}$ is well formed.

The Usual Binary Operations

Binary union and intersection (of 2 sets belonging to a universe of discourse):

- $A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$
- $A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$

U is the universe of discourse. That is, $A \subseteq U$ and $B \subseteq U$.

Union and Intersection of a set of sets

- Union: $\bigcup_{i \in I} D_i \;=\; \{x \in U \mid \;\exists\, i \in I.\, x \in D_i\}$
- Intersection: $\bigcap_{i \in I} D_i \;=\; \{x \in U \mid \;\forall\, i \in I.\, x \in D_i\}$

I is called the index set.

Example:

$I = \{0, 1, \ldots, 26\}$

Let D_i be the set of the first i letters of the alphabet, eg, $D_4 = \{a, b, c, d\}$

Then $\bigcup_{i \in I} D_i \;=\; \{a, b, \ldots, z\}$ and $\bigcap_{i \in I} D_i \;=\; \varnothing$

And if $J = \{2, 4, 7\}$, then

$\bigcup_{i \in J} D_i \;=\; \{a, b, c, d, e, f, g\}$ and $\bigcap_{i \in J} D_i \;=\; \{a, b\}$

Relations as Sets

Set-theoretic view: relations and functions are both sets of ordered pairs.

Relations:

- Any subset of $A \times B$ is a relation between elements of A and of B.
- That is, any $R \in \mathcal{P}(A \times B)$ is a relation between A and B.
- We often write xRy to mean $(x, y) \in R$
- The characteristic predicate P of R is given by:
  $P(x, y)$ is true iff $(x, y) \in R$
- There is a curried version of P, call it Q: $Q\ x\ y$ is true iff $(x, y) \in R$
- in Haskell, the infix notation (eg, $x < y$) works where $<$ is the curried version of the characteristic predicate

Flavours of Relations

- R is *reflexive* if, for all $x \in A$, $(x, x) \in R$. That is, xRx.
- R is *transitive* if $(xRy \wedge yRz)$ implies xRz.
- R is *anti-symmetric* if, whenever xRy and yRx then $x = y$.

Functions as Sets

Functions:

- Each function f from A to B, is also an element of $\mathcal{P}(A \times B)$, but subject to the condition that if $(x, y_1) \in f \wedge (x, y_2) \in f$ then $y_1 = y_2$.

Reference Details

Chapter 5 of Grassman and Tremblay is called *Sets and Relations.*

5.1 Sets and Set Operations. You must know this.

5.2 Tuples, Sequences and Power-sets.

5.2.1 Introduction: should know.

5.2.2 Tuples and Cartesian Products: must know.

5.2.3 Sequences and Strings: We'll get to this.

5.2.4 Power-sets: should know.

5.2.5 Types and Signatures: We'll cover this.

5.3 Relations: We'll be talking about this.

5.4 Properties of Relations: We'll be talking about this.