

Week 3 Tutorial
Natural Deduction
Solutions

1 Truth Tables

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow p$	$((p \rightarrow q) \rightarrow p) \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

p	q	$\neg q$	$\neg p$	$\neg p \wedge \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$
T	T	F	F	F	T	F	T
T	F	T	F	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	F	T	T

2 Natural Deduction in Propositional Logic

Question 2.1.1

1			p	
2			q	
3			p	R, 1
4			$q \rightarrow p$	\rightarrow -I, 2-3
5		$p \rightarrow (q \rightarrow p)$		\rightarrow -I, 1-4

Question 2.1.2

1			$p \wedge q$	
2			r	
3			q	\wedge -E, 1
4			$q \wedge r$	\wedge -I, 3, 2
5		$r \rightarrow (q \wedge r)$		\rightarrow -I, 2-4
6	$(p \wedge q) \rightarrow (r \rightarrow (q \wedge r))$			\rightarrow -I, 1-5

Question 2.1.3

1	$(p \vee q) \rightarrow q$	
2	p	
3	$p \vee q$	\vee -I, 2
4	q	\rightarrow -E, 1, 3
5	$p \wedge q$	\wedge -I, 2, 4
6	$p \rightarrow (p \wedge q)$	\rightarrow -I, 2-5

Question 2.2

1	$p \rightarrow q$	
2	$\neg q$	
3	p	
4	q	\rightarrow -E, 1, 3
5	$q \wedge \neg q$	\wedge -I, 2, 4
6	$\neg p$	\neg -I, 3-5
7	$\neg q \rightarrow \neg p$	\rightarrow -I, 2-6
8	$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$	\rightarrow -I, 1-7

Question 2.3

1	$p \vee (q \wedge r)$	
2	p	
3	$p \vee q$	\vee -I, 2
4	$q \wedge r$	
5	q	\wedge -E, 4
6	$p \vee q$	\vee -I, 5
7	$p \vee q$	\vee -E, 1, 2-3, 4-6

Question 2.4.1

1			$p \vee q$	
2			$\neg p$	
3			p	
4			$\neg q$	
5			$p \wedge \neg p$	\wedge -I, 2, 3
6			q	\neg -E, 4-5
7			q	
8			q	R, 7
9			q	\vee -E, 1, 3-6, 7-8
10			$\neg p \rightarrow q$	\rightarrow -I, 2-9
11			$(p \vee q) \rightarrow (\neg p \rightarrow q)$	\rightarrow -I, 1-10

Question 2.4.2

1		$(p \wedge q) \rightarrow r$	
2		p	
3		q	
4		$p \wedge q$	\wedge -I, 2, 3
5		r	\rightarrow -E, 1, 4
6		$q \rightarrow r$	\rightarrow -I, 3-5
7		$p \rightarrow (q \rightarrow r)$	\rightarrow -I, 2-6

1		$p \rightarrow (q \rightarrow r)$	
2		$p \wedge q$	
3		p	\wedge -E, 2
4		$q \rightarrow r$	\rightarrow -E, 1, 3
5		q	\wedge -E, 2
6		r	\rightarrow -E, 4, 5
7		$(p \wedge q) \rightarrow r$	\rightarrow -I, 2-6

3 Natural Deduction for Predicate Calculus

Question 3.1

1	$\forall x. P(x)$	
2	$a \mid P(a)$	\forall -E, 1
3	$\forall y. P(y)$	\forall -I, 2
4	$(\forall x. P(x)) \rightarrow (\forall y. P(y))$	\rightarrow -I, 1-3

Question 3.2

1	$(\forall x. P(x)) \wedge (\forall y. P(y) \rightarrow Q(y))$	
2	$\forall x. P(x)$	\wedge -E, 1
3	$\forall y. P(y) \rightarrow Q(y)$	\wedge -E, 2
4	$a \mid P(a)$	\forall -E, 2
5	$P(a) \rightarrow Q(a)$	\forall -E, 3
6	$Q(a)$	\rightarrow -E, 5, 4
7	$\forall z. Q(z)$	\forall -I, 6

Question 3.3

1	$\exists x. (P(x) \vee Q(x))$	
2	$a \mid P(a) \vee Q(a)$	
3	$\mid P(a)$	
4	$\mid \exists x. P(x)$	\exists -I, 3
5	$\mid (\exists x. P(x)) \vee (\exists x. Q(x))$	\vee -I, 4
6	$\mid Q(a)$	
7	$\mid \exists x. Q(x)$	\exists -I, 6
8	$\mid (\exists x. P(x)) \vee (\exists x. Q(x))$	\vee -I, 7
9	$(\exists x. P(x)) \vee (\exists x. Q(x))$	\vee -E, 2, 3-5, 6-8
10	$(\exists x. P(x)) \vee (\exists x. Q(x))$	\exists -E, 1, 2-9

Question 3.4

1	$\forall y. \exists x. P(x, y)$	
2	$b \mid \exists x. P(x, b)$	$\forall\text{-E, 1}$
3	$a \mid P(a, b)$	
4	$\mid P(a, b)$	$R, 3$
5	$P(a, b)$	-- WRONG --
6	$\forall y. P(a, y)$	$\forall\text{-I, 5}$
7	$\exists x. \forall y. P(x, y)$	$\exists\text{-I, 6}$

- The guard variable a at line (3) prevents us from moving between steps (4) and (5).
- Line (5) violates the (a is not free in q) clause in the rule.

$$\frac{\exists x. P(x) \quad \begin{array}{c} [P(a)] \\ \vdots \\ q \end{array} \quad (a \text{ arbitrary})}{q \quad (a \text{ is not free in } q)}$$

Question 3.4 Another attempt:

1	$\forall y. \exists x. P(x, y)$	
2	$\exists x. P(x, b)$	$\forall\text{-E, 1}$
3	$a \mid P(a, b)$	
4	$\mid \forall y. P(a, y)$	-- WRONG --
5	$\exists x. \forall y. P(x, y)$	$\exists\text{-I, 4}$
6	$\exists x. \forall y. P(x, y)$	$\exists\text{-E, 2, 3-5}$

- Here step (4) is wrong, as at this point, the variable b is not arbitrary: it appears in the assumption $P(a, b)$ at step (3).
- According to the other examples of $\forall\text{-I}$, we should have a vertical bar labelled b , alongside lines preceding step (3), which would be indented (this must include all the lines containing b). How could you do that in this case?

Question 4 (some)

1	($\neg p \rightarrow q$)	
2	$\neg(p \vee q)$	
3	$\neg p$	$\neg\vee\text{-E, 2}$
4	$\neg q$	$\neg\vee\text{-E, 2}$
5	q	$\rightarrow\text{-E, 1, 3}$
6	$q \wedge \neg q$	$\wedge\text{-I, 5, 4}$
7	$p \vee q$	$\neg\text{-E, 2-6}$

1	($\exists x. P(x) \wedge (\forall y. P(y) \rightarrow Q(y))$)	
2	$\exists x. P(x)$	$\wedge\text{-E, 1}$
3	$\forall y. P(y) \rightarrow Q(y)$	$\wedge\text{-E, 2}$
4	a $P(a)$	
5	$P(a) \rightarrow Q(a)$	$\forall\text{-E, 3}$
6	$Q(a)$	$\rightarrow\text{-E, 5, 4}$
7	$\exists z. Q(z)$	$\exists\text{-I, 6}$
8	$\exists z. Q(z)$	$\exists\text{-E, 2, 4-7}$