Department of Computer Science, Australian National University
COMP2600 — Formal Methods in Software Engineering
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Week 3 Tutorial

Natural Deduction

1 Truth Tables for Propositional Logic

Prove the following tautologies using truth tables

- $((p \rightarrow q) \rightarrow p) \rightarrow p$
- $\neg(p \vee q) \leftrightarrow (\neg\, p \wedge \neg\, q)$

2 Natural Deduction in Propositional Logic

2.1 Warm-up

Construct a natural deduction proof for the following alleged propositional logic theorems. Use *only* the rules of natural deduction.

1. $p \rightarrow (q \rightarrow p)$
2. $(p \wedge q) \rightarrow (r \rightarrow (q \wedge r))$
3. $\dfrac{((p \vee q) \rightarrow q)}{(p \rightarrow (p \wedge q))}$

These problems are easy because the form of the expression in each case suggests a rule to apply.

2.2 Prove: $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

This is a well-known theorem of logic and you should recognize that it is used in everyday reasoning. For present purposes, there is a simple proof that is a good illustration of $\neg$-I.

2.3 Prove: $\dfrac{p \vee (q \wedge r)}{p \vee q}$

This is a theorem which is easily proved using $\vee$-E.

2.4 Easy and Hard?

You might want to reflect on the suggestion at the start of the section that some rules are easy. Is it correct?

To help with the discussion, compare the proofs of the following formulae. The first involves $\vee$-E and one of the negation rules, while the second is much longer but almost mechanical.

1. $(p \vee q) \rightarrow (\neg p \rightarrow q)$
2. $(p \wedge q \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$ So you need to prove $\dfrac{p \wedge q \rightarrow r}{p \rightarrow (q \rightarrow r)}$ and $\dfrac{p \rightarrow (q \rightarrow r)}{p \wedge q \rightarrow r}$

3 Natural Deduction for Predicate Calculus

3.1 Prove: $(\forall x.P(x)) \rightarrow (\forall y.P(y))$

Even the simplest Predicate Calculus facts must be proved, such as change of dummy variable.

3.2 Prove: $\dfrac{(\forall x.P(x)) \wedge (\forall y.P(y) \rightarrow Q(y))}{\forall z.Q(z)}$

This is the quantified version of *modus ponens*. Again one only needs instantiation ($\forall$-E) and generalization ($\forall$-I).

3.3 Prove: $\dfrac{\exists x.(P(x) \vee Q(x))}{(\exists x.P(x)) \vee (\exists x.Q(x))}$

Working with the existential quantifier is trickier but there's a strong similarity.

3.4 Prove: $\dfrac{\forall y.\exists x.P(x, y)}{\exists x.\forall y.P(x, y)}$

This example was mentioned at the end of the last lecture.
Is it true? If so, prove it. If not, then explain why you can't.

4 Extra time

- The discussion in lectures about proving something of the form $p \vee q$, was followed by the example $\dfrac{\neg p \rightarrow q}{p \vee q}$. Try this example: in doing it you may use the basic rules given in lectures (and on the next page) and also the derived rules

$$(\neg \vee I) \quad \frac{\neg(p \vee q)}{\neg p} \qquad (\neg \vee I) \quad \frac{\neg(p \vee q)}{\neg q}$$

- The premise and conclusion of Question 3.3 are actually equivalent. Prove also the converse.
- If you change both occurrences of $\vee$ in Question 3.3 to $\wedge$ then it is also provable — prove it. In that case the converse is not provable. What goes wrong when you try to prove it? Find a counterexample.
- In Question 3.2 you can change the $\forall$ to $\exists$ in the conclusion and at one (but not both) occurrences in the premise. Prove these alternative versions.
- Most of the standard equivalences in table 2.4 of Grassmann are also straightforward. Try some if time permits.

5 Appendix: Natural Deduction Rules

5.1 Propositional Calculus - Truth Tables

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$\neg p$	$p \leftrightarrow q$
T	T	T	T	T	F	T
T	F	T	F	F	F	F
F	T	T	F	T	T	F
F	F	F	F	T	T	T

5.2 Propositional Calculus

$$(\wedge I) \quad \frac{p \qquad q}{p \wedge q} \qquad\qquad (\wedge E) \quad \frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q}$$

$$(\vee I) \quad \frac{p}{p \vee q} \qquad \frac{p}{q \vee p} \qquad\qquad (\vee E) \quad \frac{p \vee q \qquad \begin{matrix}[p] \\ \vdots \\ r\end{matrix} \qquad \begin{matrix}[q] \\ \vdots \\ r\end{matrix}}{r}$$

$$(\to I) \quad \frac{\begin{matrix}[p] \\ \vdots \\ q\end{matrix}}{p \to q} \qquad\qquad (\to E) \quad \frac{p \qquad p \to q}{q}$$

$$(\neg I) \quad \frac{\begin{matrix}[p] \\ \vdots \\ q \wedge \neg q\end{matrix}}{\neg p} \qquad\qquad (\neg E) \quad \frac{\begin{matrix}[\neg p] \\ \vdots \\ q \wedge \neg q\end{matrix}}{p}$$

5.3 Predicate Calculus

$$(\forall I) \quad \frac{P(a) \qquad (a \text{ arbitrary})}{\forall x.\ P(x)}$$

$$(\forall E) \quad \frac{\forall x.\ P(x)}{P(a)}$$

$$(\exists I) \quad \frac{P(a)}{\exists x.\ P(x)}$$

$$(\exists E) \quad \frac{\exists x.\ P(x) \qquad \begin{matrix}[P(a)] \\ \vdots \\ q\end{matrix} \qquad (a \text{ arbitrary})}{q \quad (a \text{ is not free in } q)}$$