

Computer Science COMP3600/COMP6466 Answers to Tutorial One

Following are some rough answers to the problems for Tutorial 1. The relevant sections of the textbook should be consulted for more information.

Question 1.

(i) Review the definitions of the basic notations such as Θ , O , Ω , o , ω .

See notes for Lecture 2.

(ii) Rank the following functions in the order of growth:

$n^2 \lg n$, $n^{3/2} \lg^4 n$, $3^{0.5n}$, $\lg^* n \lg^2 n$, $n^{1.44}$, $\sqrt{n^{1/3} \lg n}$, $n^{\lg \lg n}$, \sqrt{n} , n^5 .

$\lg^* n \lg^2 n$, $\sqrt{n^{1/3} \lg n}$, \sqrt{n} , $n^{1.44}$, $n^{3/2} \lg^4 n$, $n^2 \lg n$, n^5 , $n^{\lg \lg n}$, $3^{0.7n}$

The most important principles are that logarithmic functions grow slower than powers, and powers grow slower than exponential functions. To see the relative order of $n^{\lg \lg n}$ and $3^{0.5n}$, take their logarithms.

Question 2.

Review the various techniques for bounding summations.

See notes for Lecture 4.

Estimate these using the $\Theta()$ notation:

$$\sum_{k=1}^n \frac{k^2}{7^k}$$

If the ratio between adjacent terms is always greater than a constant greater than 1, or always less than a constant less than 1, the series has geometric type and the sum is $\Theta(\text{largest term})$. In this case,

$$\frac{\text{term } k+1}{\text{term } k} = \frac{(k+1)^2/7^{k+1}}{k^2/7^k} = \frac{1}{7} \left(\frac{k+1}{k} \right)^2 \leq \frac{4}{7},$$

so the sum is $\Theta(\text{largest term}) = \Theta(1)$.

$$\sum_{k=2}^n \frac{k^{1/2}}{\lg k}$$

In the case of slowly-changing terms, such as terms of polynomial size, bound by maximum and minimum terms. For an upper bound, the number of terms times the maximum term:

$$\sum_{k=2}^n \frac{k^{1/2}}{\lg k} \leq n \times \frac{n^{1/2}}{\lg n} = \frac{n^{3/2}}{\lg n}.$$

For a lower bound, half the number of terms times the minimum term in the largest half of the terms. We will use terms $k = \lceil n/2 \rceil \dots n$:

$$\sum_{k=2}^n \frac{k^{1/2}}{\lg k} \geq \frac{n}{2} \times \frac{(n/2)^{1/2}}{\lg(n/2)} \geq \frac{1}{4} \frac{n^{3/2}}{\lg n}.$$

So the value of the sum is $\Theta(n^{3/2}/\lg n)$.

$$\sum_{k=1}^n k^{7/2}$$

$\sum_{k=1}^n k^{7/2} < \sum_{k=1}^n n^{7/2} = n^{9/2}$, thus, $\sum_{k=1}^n k^{7/2} = O(n^{9/2})$. Also, $\sum_{k=1}^n k^{7/2} \geq \sum_{k=n/2}^n k^{7/2} \geq \sum_{k=n/2}^n (n/2)^{7/2} = (n/2)^{9/2}$, i.e, $\sum_{k=1}^n k^{7/2} = \Omega(n^{9/2})$. Thus,

$$\sum_{k=1}^n k^{7/2} = \Theta(n^{9/2}).$$

Question 3. Review the two approaches for solving recurrences, and apply them to solve the following recurrences.

See notes for Lecture 5.

- (a) $T(n) = T(n/3) + n^{5/4}$.

We apply the iteration method (apply the recurrence to itself repeatedly). It is obvious that $k = \log_3 n$ when $\frac{n}{3^k} = 1$. Then,

$$\begin{aligned} T(n) &= T(n/3) + n^{5/4} \\ &= T(n/3^2) + n^{5/4} + (n/3)^{5/4} \\ &= \dots \\ &= T(1) + n^{5/4} + (n/3)^{5/4} + (n/3^2)^{5/4} + \dots + (n/3^k)^{5/4}. \end{aligned}$$

This is a series of geometric type so its sum is $\Theta(n^{5/4})$. The term $T(1)$ is negligible in comparison, so the answer is $T(n) = \Theta(n^{5/4})$.

- (b) $T(n) = 3T(n/2) + 4n$.

Apply the iteration method again. This time $k = \log_2 n$ steps are needed.

$$\begin{aligned}
 T(n) &= 3T(n/2) + 4n \\
 &= 3^2T(n/2^2) + 3 * 4 * n/2 + 4n \\
 &= 3^3T(n/2^3) + (3/2)^2 4n + (3/2)4n + 4n \\
 &= \dots \\
 &= 3^k T(1) + (3/2)^{k-1} 4n + \dots + (3/2)4n + 4n
 \end{aligned}$$

The series, apart from the term $3^k T(1)$, is of geometric type so its sum is $\Theta(3^k)$, which is also the size of $3^k T(1)$. Also note that for $k = \log_2 n$, $3^k = n^{\log_2 3}$. So the answer is $T(n) = \Theta(n^{\log_2 3})$.

- (c) $T(n) = T(\lfloor \sqrt{n} \rfloor) + n \log n$.

We use mathematical induction, starting with an educated guess that the answer might be $T(n) = \Theta(n \lg n)$. To avoid having either n or $\lfloor \sqrt{n} \rfloor$ equal to 1 (which has a logarithm of 0), we start the induction at $n = 4$.

Assume that for some constant $c > 1$, and all $n \geq 5$, we have

$$T(n') \leq cn' \log n'$$

for $4 \leq n' < n$. (This is the induction hypothesis.) Then, applying the recurrence, we have

$$\begin{aligned}
 T(n) &= T(\lfloor \sqrt{n} \rfloor) + n \log n \\
 &\leq c\sqrt{n} \log \sqrt{n} + n \log n \\
 &\leq \left(\frac{1}{2}cn^{-1/2} + 1\right) n \log n
 \end{aligned}$$

For $n \geq 4$, we have $\frac{1}{2}cn^{-1/2} + 1 \leq c$ provided c is large enough, for example when $c = 4$. Therefore, by induction, $T(n) \leq 2n \log n$ for $n \geq 4$.

We also have $T(n) \geq n \log n$ obviously (look at the recurrence), so the answer is $T(n) = \Theta(n \log n)$.

Question 4. In the linear-time selection algorithm, the elements in the input sequence are divided into groups of size 5. Does the algorithm still run in linear time if the input elements are divided into groups of size 17 instead?

Yes, when the group size is 17, we can easily bound the sizes of R_1 and R_3 , using the similar discussion as we have done for group size 5.

We assume that the i th smallest element that we are looking for is in R_3 , we now show that the size of R_1 is bounded. Otherwise (i.e., the element is in R_1 or R_2), we can show that the size of R_3 is bounded as well.

$|R_1| \geq 9\left(\frac{\lceil n/17 \rceil}{2} - 2\right) \geq \frac{9n}{34} - 18$. Thus, the size of R_3 is $|R_3| = n - |R_1| - |R_2| \leq n - |R_1| \leq \frac{25n}{34} + 18$. The time complexity is thus represented as the following recurrence.

$$T(n) \leq \begin{cases} \Theta(1) & \text{if } n \leq 100 \\ T(\lceil n/17 \rceil) + T(25n/34 + 18) + O(n) & \text{if } n > 100 \end{cases}$$

We will prove by the method of substitution that $T(n) \leq cn$ for some constant c . Choose c large enough that $T(n) \leq cn$ for $n \leq 100$.

Also suppose the $O(n)$ term in the recurrence is bounded by an for $n > 100$.

For $n > 100$, the recurrence says

$$T(n) \leq T(\lceil n/17 \rceil) + T(25n/34 + 18) + an \tag{1}$$

$$\leq c\lceil n/17 \rceil + 25cn/34 + 18c + an \tag{2}$$

$$\leq cn/17 + c + 25cn/34 + 18c + an \tag{3}$$

$$= 27cn/34 + 19c + an \tag{4}$$

$$= cn + (-7cn/34 + 19c + an) \tag{5}$$

The expression $-7cn/34 + 19c + an$ is negative if $n > 100$ and $c \geq 102a$.

Therefore, for some c , $T(n) \leq cn$ always.