The α-β lemma

Let Σ be a finite alphabet. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ and $\beta_1, \beta_2, \beta_3, \ldots$ be infinite sequences of strings over Σ.

Suppose the language $L \subseteq \Sigma^*$ satisfies the following properties:

(a) For all i, $\alpha_i\beta_i \in L$.

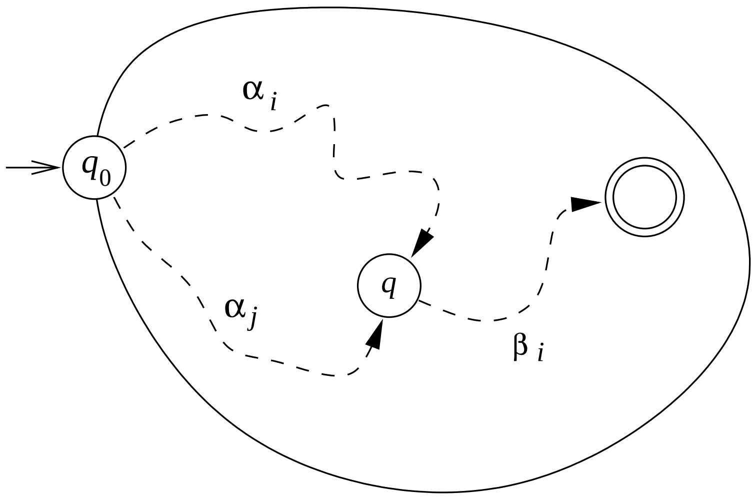
(b) For all i, j with $i \neq j$, $\alpha_j\beta_i \notin L$.

Then L is not a regular language.

Proof. Suppose that L is regular. Then there is a DFA $A = (Q, \Sigma, \delta, q_0, F)$ that accepts L. Let $\hat{\delta}$ be the extended transition function for A.

(Recall: for any state $q \in Q$ and string $x \in \Sigma^*$, $\hat{\delta}(q, x)$ is the state that A reaches if it is in state q and then string x is input. Thus $L = \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) \in F\}$.)

Since A is finite but the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ is infinite, there are two members of the sequence, say α_i and α_j for $i \neq j$, and a state q, such that $\hat{\delta}(q_0, \alpha_i) = \hat{\delta}(q_0, \alpha_j) = q$. According to condition (a), $\alpha_i\beta_i$ is accepted, so $\hat{\delta}(q, \beta_i) \in F$. But then $\alpha_j\beta_i$ is accepted as well, which contradicts condition (b).



Since α_i and α_j lead to the same state, and $\alpha_i\beta_i$ is accepted, $\alpha_j\beta_i$ must be accepted as well.

Note. Nobody said that the 'α's and 'β's are in L. Actually, they usually aren't.

Author: Brendan McKay. March 7, 2007. But this is not my invention.