Formalised Proof Theory of Bi-Intuitionistic Logic

COMP3740
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Outline

- Motivation
- Semantics
- Sequents
- Nested Sequents
- Linear Nested Sequents
- Intuition behind the Calculus
- Tau translations
- Soundness
- Formalising in Coq
- Current Progress
- Future Work
Bi-Int logic allows for forward and backward movement between worlds or states.

Previous set of rules proposed are inefficient and grew exponentially as the size of formulae increased.

We believe that we can get away with a lot of the complex machinery to present a simpler and elegant solution.
Bi-Int logic is an extension to classical logic. Now we have a notion of worlds and a reachability relation among these worlds.

- \( w \models p \) iff \( p \in I(w) \);
- \( w \models \top \) always; \( w \not\models \bot \) never;
- \( w \models A \land B \) iff \( w \models A \) and \( w \models B \); \( w \models A \lor B \) iff \( w \models A \) or \( w \models B \);
- \( w \models A \supset B \) iff, for any \( w' \geq w \), \( w' \not\models A \) or \( w' \models B \);
- \( w \models A \prec B \) iff, for some \( w' \leq w \), \( w' \models A \) and \( w' \not\models B \).

Fig 1: Semantics of Bilnt Logic (Pinto & Uustalu, 2009)
A Sequent is a relation between a set of premises and a set of conclusions.

\[ \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \]

Intuitively, we think of a valid sequent as “If everything on the left of the sequent is true then something on the right must be true”
Nested Sequents

It is a finite tree whose nodes are labelled with Sequents.

They model the tree structure of the reachability relation.

However, they grow exponentially in size.

Can we do better?
Linear Nested Sequents

We restrict the tree structure to a single, linear branch. This helps us achieve linear growth.

Known to work for tense logic (Gore & Lellman, 2019).
The northeast arrow says “In every successor world”

“For an arbitrary world w, If A -> B is true in every successor world(x) of w, then either A is false or B is true in every successor worlds of x.”
This rule can be said to be true in any world regardless of our knowledge of previous worlds.

As such, we can add an arbitrary context (G) as long as it is uniform between the premises and the conclusion.

\[
\begin{align*}
G \uparrow \Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi, A \rightarrow B \quad &\Rightarrow A \Rightarrow B \\
G \uparrow \Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi, A \rightarrow B
\end{align*}
\]

G represents many worlds and our knowledge of those worlds so far.
Theorems like Soundness and Completeness do not depend on the structure of the rules, but rather on the meaning behind them.

Tau translations take in a premise (or conclusion) and convert it into a Bi-Int formula.

\[ \tau(\Gamma \Rightarrow \Delta \uparrow G) = \hat{\Gamma} \rightarrow (\check{\Delta} \lor \tau(G)) \]
Soundness

If something can be proven with our calculus then it can be explained using the semantics.

What is this something? It is a sequence of rules applied one after the other.

If the rules are provable then the rules can be explained using the semantics.

If the tau translations of the premises are valid, then the tau translation of the conclusion is valid.
Formalising using Coq

Coq is a language used to verify proofs.

Written proofs can trivialise some points that the machine would need a non-trivial proof for.

Previously, it was affirmed that “something being provable” can be simplified to “rules are provable” as we are working with a sequence of rules.

How does Coq know this? It doesn't understand it by itself. We need a proof for it.

This proof can be done using Induction.
Current Progress

I have managed to implement everything from the semantics to the tau translations.

However, when proving soundness I am stuck.

\[
\frac{G \vdash \Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi, A \rightarrow B \uparrow A \Rightarrow B}{G \vdash \Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi, A \rightarrow B}
\]

I am unsure of how to show that G represents knowledge of previous worlds and the relations between them.
Future Work

1. Complete the Soundness Proof
2. Prove Completeness.
Thanks for listening Q(^_^Q)
And ... Questions?