Due date: Monday, August 26, 2019, 9:00

Late penalty: 5% per each day or part thereof.
Submission: via box marked COMP4630 on the ground floor of the CSIT Building 108.
Worth: 1/6 of 50% of the final mark ie 8.33% of the final mark.
Full marks for formulations of correct answers that clearly show the steps to obtain the solution.

Question 1 (FOL, 3+3+4 pts). Formulate the following statements in first-order logic. Use obvious predicates, such as in dog(x) to express that “x is a dog”.

1. Every white dog chases every black cat.
2. There is no dog that does not chase some cat.
3. Pluto, a dog, chases every dog that does not chase Pluto.
   For the curious (not marked): what can be said about whether Pluto is chasing himself?
   Hint: converting your formula to CNF and using the Resolution calculus might help.

Answer

1. ∀x. ((dog(x) ∧ white(x)) → (∀y. ((cat(y) ∧ black(y)) → chases(x, y)))
2. ¬∃x. (dog(x) ∧ ¬∃y. (cat(y) ∧ chase(x, y))
3. dog(Pluto) ∧ ∀x. ((dog(x) ∧ ¬chase(x, Pluto)) → chase(Pluto, x))
   This formula is easily converted into CNF. One derives with one resolution step and one factoring step chase(Pluto, Pluto).
   That is, from what was given it follows that Pluto chases himself! That was not obvious from the English formulation.

Question 2 (Semantics of FOL, 5 + 5 + 8 pts). Let $F = (P(a) ∧ (∀x. P(x) → P(f(x))))$ be a first-order logic formula and $I$ an interpretation. Which of the following statements is true? false? In each case provide a proof by arguing with definitions related to the semantics of first-order.

1. If $I \models F$ then $I \models ∃x. P(x)$
2. If $I \models F$ then $I \models ∀x. P(x)$
3. If $I \models F$ then for every ground term $t \in \{a, f(a), f(f(a)), \ldots\}$ it holds $I \models P(t)$.

Answer

(1) The statement is true. Proof: assume $I \models F$ for some interpretation $I = (D_I, α_I)$. It follows $I \models P(a)$, and hence $I \models \{x → α_I[a]\} \models P(x)$. By semantics of quantifiers $I \models ∃x. P(x)$.

(2) The statement is false. Proof: Take the interpretation $I = (D_I, α_I)$ such that
   
   $D_I = \mathbb{N}$
   $α_I[a] = 0$
   $α_I[f] = x ↦ x + 2$
   $α_I[P] = x ↦ x \text{ is even}$

   Clearly, $I \models F$ but $I \not\models ∀x. P(x)$ as $I \models \{x ↦ 1\} \not\models P(x)$. 

(3) The statement is true. Proof: assume \( I \models F \) for some interpretation \( I = (D_I, a_I) \). Let \( t = f^n(a) \) be an input ground term wrt. \( F \), for some \( n \geq 0 \) \( f^n = (f \cdots f(a)) \).

We prove the statement by induction on \( n \).

**Induction start.** If \( n = 0 \) then \( P(t) = P(f^0(a)) = P(a) \) and \( I \models P(a) \) follows immediately from \( I \models F \).

**Induction step.** If \( n > 0 \) then \( I \models P(f^{n-1}(a)) \) by induction. From \( I \models F \) it follows \( I \models \forall x. P(x) \to P(f(x)) \). By semantics of quantifiers and of \( \to \) conclude \( I \models P(f^n(a)) \), equivalently \( I \models P(f^n(a)) \).

**Question 3** (Tableau calculus, 14 pts). Apply the tableau calculus to prove that the formula \( (\forall x. P(x) \to Q(x)) \to ((\forall x. P(x)) \to (\forall x. Q(x))) \) is valid.

**Answer**

Let us assume that the given formula is not valid.

1. \( I \not\models (\forall x. P(x) \to Q(x)) \to ((\forall x. P(x)) \to (\forall x. Q(x))) \) (assumption)
2. \( I \models \forall x. P(x) \to Q(x) \) (by 1 and \( \to \))
3. \( I \not\models (\forall x. P(x)) \to (\forall x. Q(x)) \) (by 1 and \( \to \))
4. \( I \models \forall x. P(x) \) (by 3 and \( \to \))
5. \( I \not\models \forall x. Q(x) \) (by 3 and \( \to \))
6. \( I \not\models Q(a) \) (by 5 and \( \forall (x \mapsto a \text{ fresh}) \))
7. \( I \models P(a) \to Q(a) \) (by 2 and \( \forall (x \mapsto a) \))

We have two cases:

8a. \( I \models P(a) \) (by 7 and \( \to \))
8b. \( I \models Q(a) \) (by 7 and \( \to \))

9a. \( I \models P(a) \) (by 4 and \( \forall (x \mapsto a) \))
9b. \( \bot \) (by 6 and 8b)

**Question 4** (Tableau calculus completeness, 6 + 14 pts). Let \( O \) be the conjunction of the following formulas, axiomatizing dense strict orderings without right endpoints.

\[
\begin{align*}
(1) &\quad \forall x. \neg(x < x) \\
(2) &\quad \forall x. \forall y. \forall z. (x < y \land y < z \to x < z) \\
(3) &\quad \forall x. \forall y. (x < y \to \exists z. (x < z \land z < y)) \\
(4) &\quad \forall x. \exists y. (x < y)
\end{align*}
\]

1. Extend the tableau calculus with rules for “<” such that a formula \( O \to F \) is valid iff there is a closed tableaux for \( F \). No proof is required at this stage.

2. Prove the completeness of the extended calculus. You can take the proof presented in class as a basis and only state the necessary changes.

**Answer**

(1) The new rules for < are as follows (\( a \) is a constant):

\[
\begin{align*}
(1) &\quad I \not\models t < t \quad \text{for any term } t \\
(2) &\quad I \models s < t \quad \text{for any terms } s, t \text{ and } u \\
(3) &\quad I \models s < a \quad \text{for any terms } s, t \text{ and fresh } a \\
(4) &\quad I \models t < a \quad \text{for any term } t \text{ and fresh } a
\end{align*}
\]

The rules (1) - (4) correspond to the axioms (1)-(4). The rule (1) has the same effect as adding axiom (1) in the form of the statement “\( I \models \forall x. \neg(x < x) \)” to the tableau proof and applying to it the old inference rules. Notice, though, that the inference rule will skip adding statements of the form “\( I \models \neg(t < t) \)”.

The rule (2) corresponds to the transitivity axiom (2). Notice that it applies only to already derived inequalities \( I \models s < t \) and \( I \models t < u \) but cannot derive instances of the axiom (2) itself.

Rules (3) and (4) are explained similarly and build in a correct treatment of the embedded \( \exists \)-quantifier.

(2) The **fairness** assumption is extended to the new rules (1) - (4) as follows.
• Rule (1) is to be applied for all ground terms \( t \).
• Rule (2) is to be applied whenever possible.
• Rule (3) is applied once for every statement “\( I \models s < t \)” on the branch.
• Rule (4) is applied once for every ground term \( t \).

We assume the same setup as in the completeness proof. Hence, let \( F \) again be a formula without free variables. Assume a fair tableau with the statement “\( I \models \neg F \)” in the root and that has an open branch \( P \). According to question 4.1, we then have to show there is an interpretation \( I \) such that \( I \models \neg O \rightarrow F \). Equivalently, we will show \( I \models O \) and \( I \models \neg F \).

Item (1) of completeness proof: The extended fairness assumption provides in \( P \) an extended Hintikka set, which has all properties of a Hintikka set and additionally satisfied the following properties:

- \((I \models \neg t < t) \in P\) for every ground term \( t \).
- \((I \models s < t) \in P \) and \((I \models t < u) \in P\) implies \((I \models s < u) \in P\).
- \((I \models s < t) \in P\) implies \((I \models s \neq a) \in P\) and \((I \models a < t) \in P\), for some fresh constant \( a \).
- For every ground term \( t \) there is a fresh constant \( a \) such that \((I \models t \neq a) \in P\).

Items (2) and (3) are unchanged. For the predicate symbol \(<\) the interpretation \( I \), thus, is defined as follows:

\[
\alpha_I(<)(s, t) = \begin{cases} 
\text{true} & \text{if } (I \models s < t) \in P \\
\text{false} & \text{otherwise}
\end{cases}
\]

Because every extended Hintikka set is also a Hintikka set as defined in the given completeness proof, item (4) of that proof still holds and it follows \( I \models \neg F \), the statement in the root of the tableau.

To complete the proof, only \( I \models O \) remains to be shown, the “extended property (4)”. We can consider the four axioms separately.

Regarding axiom (1), by extended Hintikka set \((I \models \neg t < t) \in P\) for every ground term \( t \). With the definition of \( I \) it follows \( I \models \neg t < t \). By semantics of first-order logic, \( I \models \neg(t < t) \). Finally, as the domain of \( I \) consists of all ground terms (only), \( I \models \forall x. \neg(x < x) \).

Regarding axiom (2), assume by way of contradiction, that \( I \models \neg \forall x. \forall y. \exists z. (x < y \land y < z \rightarrow x < z) \). That is, there are ground terms \( s, t \) and \( u \) such that \( I \models s < t \) and \( I \models t < u \) but \( I \models s < u \). (This argumentation is somewhat shortcutting.) With the definition of \( I \) it follows \((I \models s < t) \in P \) and \((I \models t < u) \in P \). By extended Hintikka set we get \((I \models s < u) \in P \). By definition of \( I \) again \( I \models s < u \), a direct contradiction.

Regarding axiom (3), assume by way of contradiction, that \( I \models \neg \forall x. \forall y. (x < y \rightarrow \exists z. (x < z \land z < y)) \). That is, there are are ground terms \( s \) and \( t \) such that \( I \models s < t \) but \( I \models \exists z. (s < z \land z < t) \). With the definition of \( I \) it follows \((I \models s < t) \in P \). By extended Hintikka set we get \((I \models s < a) \in P \) and \((I \models a < t) \in P \), for some fresh constant \( a \). Because \( a \) evaluates to \( a \) under \( I \) it follows \( I \models \{z \mapsto a\} s < z \land z < t \). By semantics of first-order logic, \( I \models \exists z. (s < z \land z < t) \), a direct contradiction.

The proof for axiom (4) is similar to the proof for axiom (3) and is omitted.

A final note: The rule (1) can be replaced by the weaker rule

\[
\frac{I \models t < t}{\bot}
\]

for any term \( t \).

The proof above requires a slight modification for the case of axiom (1).

**Question 5** (Substitutions, 10 pts). Proof by induction on the term structure: if \( \sigma \) is a substitution and \( t \) is a term such that \( \text{dom}(\sigma) \cap \text{var}(t) = \emptyset \) then \( t\sigma = t \).

**Answer**

Let \( \sigma \) be a substitution and \( t \) a term. Assume that \( \text{dom}(\sigma) \cap \text{var}(t) = \emptyset \). We prove by induction on the structure of \( t \) that \( t\sigma = t \).
• **Base cases:** We have two base cases here:
  
  - Assume that \( t := x \) for some variable \( x \). We need to show that \( x\sigma = x \). A \( x \in \text{var}(t) \) and \( \text{dom}(\sigma) \cap \text{var}(t) = \emptyset \) we know that \( x \notin \text{dom}(\sigma) \), hence \( x\sigma = x \) by definition of substitutions.
  
  - Assume that \( t := c \) for some constant \( c \). We need to show that \( c\sigma = c \). But we already know that \( c\sigma = c \) by definition of substitutions.

• **Inductive case:** We have only one inductive case here. Assume that \( t := f(t_1, t_2, \ldots, t_n) \) for some terms \( t_1, \ldots, t_n \) and an \( n \)-ary function symbol \( f \). Our induction hypothesis is: \( t_i\sigma = t_i \). By definition of substitutions we have that \( t\sigma = (f(t_1, \ldots, t_n))\sigma = f(t_1\sigma, \ldots, t_n\sigma) \). But as by induction hypothesis we get \( t_i\sigma = t_i \) for \( i \in \{1, \ldots, n\} \), then we get that \( f(t_1\sigma, \ldots, t_n\sigma) = f(t_1, \ldots, t_n) \). So we have that \( t\sigma = t \).

**Question 6** (Unification, 8 + 8 pts). Apply the rule based unification algorithm to the unification problems \( E \) below and read off the result, i.e., either \( \bot \) or the mgu (\( a \) is a constant, \( x, y \) and \( z \) are variables):

1. \( E = \{ y \doteq x, \ g(f(a, y)) \doteq g(x) \} \)
2. \( E = \{ g(f(x, y)) \doteq g(z), \ g(f(y, a)) \doteq g(z) \} \)

**Answer**

(1)

\[
\begin{align*}
E : & \quad y \doteq x, \ g(f(a, y)) \doteq g(x) \quad \text{(given)} \\
E_1 : & \quad y \doteq x, \ f(a, y) \doteq x \quad \text{(by Decompose)} \\
E_2 : & \quad y \doteq x, \ x \doteq f(a, y) \quad \text{(by Orient)} \\
E_3 : & \quad y \doteq f(a, y), \ x \doteq f(a, y) \quad \text{(by Apply } x \mapsto f(a, y)\text{)} \\
E_4 : & \quad \bot \quad \text{(by Occur Check)}
\end{align*}
\]

There is no unifier of \( E \).

(2)

\[
\begin{align*}
E : & \quad g(f(x, y)) \doteq g(z), \ g(f(y, a)) \doteq g(z) \quad \text{(given)} \\
E_1 : & \quad f(x, y) \doteq z, \ g(f(y, a)) \doteq g(z) \quad \text{(by Decompose)} \\
E_2 : & \quad z \doteq f(x, y), \ g(f(y, a)) \doteq g(z) \quad \text{(by Orient)} \\
E_3 : & \quad z \doteq f(x, y), \ g(f(y, a)) \doteq g(f(x, y)) \quad \text{(by Apply } z \mapsto f(x, y)\text{)} \\
E_4 : & \quad z \doteq f(x, y), \ f(y, a) \doteq f(x, y) \quad \text{(by Decompose)} \\
E_5 : & \quad z \doteq f(x, y), \ y \doteq x, \ a \doteq y \quad \text{(by Decompose)} \\
E_6 : & \quad z \doteq f(x, x), \ y \doteq x, \ a \doteq x \quad \text{(by Apply } y \mapsto x\text{)} \\
E_7 : & \quad z \doteq f(x, x), \ y \doteq x, \ x \doteq a \quad \text{(by Orient)} \\
E_8 : & \quad z \doteq f(a, a), \ y \doteq a, x \doteq a \quad \text{(by Apply } x \mapsto a\text{)}
\end{align*}
\]

Result is mgu \( \sigma = \{ z \mapsto f(a, a), \ y \mapsto a, x \mapsto a \} \).

**Question 7** (First-order resolution, 12 pts). Find a Resolution refutation of the following clause set. As for the mgu’s used, it suffices to only state them, you do not need to show the details of the runs of the unification algorithm. The letter \( a \) is a constant, and \( x \) and \( y \) are variables.

(1) \( P(x) \lor Q(f(x, y)) \lor Q(f(y, x)) \)
(2) \( \neg Q(x) \lor R(x) \)
(3) \( \neg P(a) \lor \neg P(x) \)
(4) \( \neg R(f(a, y)) \)

**Answer**
1. \( P(x) \lor Q(f(x, y)) \lor Q(f(y, x)) \) (given)
2. \( \neg Q(x) \lor R(x) \) (given)
3. \( \neg P(a) \lor \neg P(x) \) (given)
4. \( \neg R(f(a, y)) \) (given)
5. \( P(y) \lor Q(f(y, y)) \) (Fact. 1, \( \{x \mapsto y\}\))
6. \( Q(f(a, a)) \lor \neg P(x) \) (Res. 5 into 3, \( \{y \mapsto a\}\))
7. \( Q(f(a, a)) \lor Q(f(y, y)) \) (Res. 6 into 5, \( \{x \mapsto y\}\))
8. \( Q(f(a, a)) \) (Fact. 7, \( \{y \mapsto a\}\))
9. \( R(f(a, a)) \) (Res. 8 into 2, \( \{x \mapsto f(a, a)\}\))
10. \( \square \) (Res. 9 into 4, \( \{y \mapsto a\}\))