Introduction to Modal and Temporal Logic

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27 August 2017 Version 1.5
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History: Logic of Necessity and Possibility

Classical logic is truth-functional: truth value of larger formula determined by truth value(s) of its subformula(e) via truth tables for $\land$, $\lor$, $\neg$, and $\to$.

Lewis 1920s: How to capture a non-truth-functional notion of “A Necessarily Implies B”? $(A \nless B)$

Take $A \nless B$ to mean “it is impossible for $A$ to be true and $B$ to be false”

Write $PA$ for “$A$ is possible” then:

$\neg PA$ is “$A$ is impossible”

$\neg P\neg A$ is “not-$A$ is impossible”

$NA := \neg P\neg A$ “$A$ is necessary”

$A \nless B := N(A \to B) = \neg P\neg (A \to B) = \neg P\neg (\neg A \lor B) = \neg P(A \land \neg B)$

Modal Logic: “possibly true” and “necessarily true” are modes of truth
Preliminaries

Directed Graph $\langle V, E \rangle$: where

$$V = \{v_0, v_1, \cdots \}$$ is a set of vertices

$$E = \{(s_1, t_1), (s_2, t_2), \cdots \}$$ is a set of edges from source vertex $s_i \in V$ to target vertex $t_i \in V$ for $i = 1, 2, \cdots$.

Cross Product: $V \times V$ stands for $\{(v, w) | v \in V, w \in V\}$ the set of all ordered pairs $(v, w)$ where $v$ and $w$ are from $V$.

Directed Graph $\langle V, E \rangle$: where $V = \{v_0, v_1, \cdots \}$ is a set of vertices and $E \subseteq V \times V$ is a binary relation over $V$.

Successor: vertex $v \in V$ is a successor of a vertex $w \in V$ if $(w, v) \in E$

Iff: means if and only if.
Logic = Syntax and (Semantics or Calculus)

Syntax: formation rules for building formulae $\varphi, \psi, \cdots$ for our logical language

Assumptions: a (usually) finite collection $\Gamma$ of formulae

Semantics: $\varphi$ is a logical consequence of $\Gamma$ $(\Gamma \models \varphi)$

Calculi: $\varphi$ is derivable (purely syntactically) from $\Gamma$ $(\Gamma \vdash \varphi)$

Soundness: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

Consistency: Both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$ should not hold for any $\varphi$

Decidability: Is there an algorithm to tell whether or not $\Gamma \models \varphi$?

Complexity: Time/space required by algorithm for deciding whether $\Gamma \models \varphi$?
Syntax of Modal Logic

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \)  \( (Atm) \)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \langle \rangle \varphi \mid \Box \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \)  \( (Fml) \)

Examples: \( \Box p_0 \rightarrow p_2 \quad \Box p_3 \rightarrow \Box \Box p_1 \quad \Box(p_1 \rightarrow p_2) \rightarrow ((\Box p_1) \rightarrow (\Box p_2)) \)

Variables: \( p, q, r \) stand for atomic formulae while \( \varphi, \psi \) possibly with subscripts stand for arbitrary formulae (including atomic ones)

Schema/Shapes: \( \Box \varphi \rightarrow \varphi \quad \Box \varphi \rightarrow \Box \Box \varphi \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

Schema Instances: Uniformly replace the formula variables with formulae

Examples: \( \Box p_0 \rightarrow p_0 \) is an instance of \( \Box \varphi \rightarrow \varphi \) but \( \Box p_0 \rightarrow p_2 \) is not

Formula Length: number of logical symbols, excluding parentheses, where \( \text{length}(p_0) = \text{length}(p_1) = \cdots = 1 \)

Example: \( \text{length}(\Box p_0 \rightarrow p_2) = 4 \)
Kripke Semantics for Modal Logic

**Motivation:** Give an intuitive meaning to syntactic symbols.

**Motivation:** Give the meaning of “ϕ is true”

**Motivation:** Define a meaning of “ϕ is a logical consequence of Γ” \( (Γ \models ϕ) \)

**Goal:** Prove some interesting properties of logical consequence.
**Kripke Semantics for Modal Logic**

**Kripke Frame:** directed graph \( \langle W, R \rangle \) where \( W \) is a non-empty set of points/worlds/vertices and \( R \subseteq W \times W \) is a binary relation over \( W \)

**Valuation:** on a Kripke frame \( \langle W, R \rangle \) is a map \( \vartheta : W \times \text{Atm} \mapsto \{t, f\} \) telling us the truth value (\( t \) or else \( f \)) of every atomic formula at every point in \( W \)

**Kripke Model:** \( \langle W, R, \vartheta \rangle \) where \( \vartheta \) is a valuation on a Kripke frame \( \langle W, R \rangle \)

**Example:** If \( W = \{w_0, w_1, w_2\} \) and \( R = \{(w_0, w_1), (w_0, w_2)\} \) and \( \vartheta(w_1, p_3) = t \) then \( \langle W, R, \vartheta \rangle \) is a Kripke model as pictured below:

\[
\begin{array}{c}
\text{R} & w_1 \\
\downarrow & \downarrow \\
w_0 & \text{R} \\
\downarrow & \downarrow \\
w_2 \\
\end{array}
\]

\begin{align*}
\vartheta(w_0, p) & = f \text{ for all } p \in \text{Atm} \\
\vartheta(w_1, p) & = f \text{ for all } p \neq p_3 \in \text{Atm} \\
\vartheta(w_2, p) & = f \text{ for all } p \in \text{Atm} \\
\vartheta(w_0, \langle \rangle p_1) & = ? \\
\vartheta(w_0, []p_1) & = ?
\end{align*}
Kripke Semantics for Modal Logic

Given some model \( \langle W, R, \vartheta \rangle \) and some \( w \in W \), we lift \( \vartheta \) to compute the truth value of a non-atomic formula by recursion on its shape:

\[
\vartheta(w, \neg \varphi) = \begin{cases} 
  t & \text{if } \vartheta(w, \varphi) = f \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, \varphi \land \psi) = \begin{cases} 
  t & \text{if } \vartheta(w, \varphi) = t \text{ and } \vartheta(w, \psi) = t \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, \varphi \lor \psi) = \begin{cases} 
  t & \text{if } \vartheta(w, \varphi) = t \text{ or } \vartheta(w, \psi) = t \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, \varphi \rightarrow \psi) = \begin{cases} 
  t & \text{if } \vartheta(w, \varphi) = f \text{ or } \vartheta(w, \psi) = t \\
  f & \text{otherwise}
\end{cases}
\]

Intuition: classical connectives behave as usual at a world (truth functional)
Kripke Semantics for Modal Logic

Given some model $\langle W, R, \vartheta \rangle$ and some $w \in W$, we lift $\vartheta$ to compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \langle \varphi \rangle) = \begin{cases} 
  t & \text{if some } v \in W \text{ has } wRv \text{ and } \vartheta(v, \varphi) = t \\
  f & \text{otherwise}
\end{cases}$$

$$\vartheta(w, [\varphi]) = \begin{cases} 
  t & \text{if every } v \in W \text{ with } wRv \text{ has } \vartheta(v, \varphi) = t \\
  f & \text{otherwise}
\end{cases}$$

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = t$ then $\langle W, R, \vartheta \rangle$ is a Kripke model as pictured below:

$$\vartheta(w_0, \langle \varphi \rangle) = t$$
$$\vartheta(w_0, [\varphi]) = f$$
$$\vartheta(w_1, [\varphi]) = t$$
$$\vartheta(w_1, [\neg \varphi]) = t$$
$$\vartheta(w_0, [\varphi]) = t$$

Intuition: truth of modalities depends on underlying $R$ (not truth functional)
Semantics: Examples

Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be any Kripke model, and $w \in W$.

**Example:** If $\vartheta(w, [\square] \varphi) = t$ then $\vartheta(w, \langle \rangle \neg \varphi) = f$

**Example:** If $\vartheta(w, \langle \neg \varphi \rangle) = f$ then $\vartheta(w, \neg \langle \neg \varphi \rangle) = t$ \hspace{1cm} $[\square] \varphi \rightarrow \neg \langle \neg \varphi \rangle$

**Example:** If $\vartheta(w, \langle \neg \varphi \rangle) = t$ then $\vartheta(w, [\neg \varphi]) = f$

**Example:** If $\vartheta(w, [\neg \varphi]) = f$ then $\vartheta(w, \neg [\neg \varphi]) = t$ \hspace{1cm} $\langle \varphi \rangle \rightarrow \neg [\neg \varphi]$

**Exercise:** Show that all these implications are reversible.

**Example:** $\vartheta(w, [\neg \varphi]) = t$ if and only if $\vartheta(w, \neg \langle \neg \varphi \rangle) = t$

**Example:** $\vartheta(w, \langle \neg \varphi \rangle) = t$ if and only if $\vartheta(w, \neg [\neg \varphi]) = t$
Classical (Two-Valued) Nature of Kripke Semantics

Lemma 1  For any Kripke model $\langle W, R, \vartheta \rangle$, any $w \in W$ and any formula $\varphi$, either $\vartheta(w, \varphi) = t$ or else $\vartheta(w, \varphi) = f$.

Proof: Pick any Kripke model $\langle W, R, \vartheta \rangle$, any $w \in W$, and any formula $\varphi$. Proceed by induction on the length $l$ of $\varphi$.

Base Case $l = 1$: If $\varphi$ is an atomic formula $p$, either $\vartheta(w, p) = t$ or $\vartheta(w, p) = f$ by definition of $\vartheta$. So the lemma holds for all atomic formulae.

Ind. Hyp. : Lemma holds for all formulae of length less than some $n > 1$.

Induction Step: If $\varphi$ is of length $n$, then consider the shape of $\varphi$.

$\varphi = \langle \rangle \psi$: If $w$ has no $R$-successors, then $\vartheta(w, \langle \rangle \psi) = f$, and $\vartheta(w, \langle \rangle \psi) = t$ is impossible by its definition. Else pick any $v \in W$ with $wRv$. By IH, either $\vartheta(v, \psi) = t$ or else $\vartheta(v, \psi) = f$ since $\psi$ is smaller than $\varphi$. Either all $R$-successors of $w$ make $\psi$ false, or else at least one of them makes $\psi$ true. Hence, either $\vartheta(w, \langle \rangle \psi) = f$ or else $\vartheta(w, \langle \rangle \psi) = t$. 
Semantic Forcing Relation $\models$ and its negation $\not\models$

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame

Let $\Gamma$ be a set of formulae, and $\varphi$ be a formula

<table>
<thead>
<tr>
<th>Forces</th>
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<th>$\bullet \not\models \varphi$</th>
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<td>in a model</td>
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<td>$\exists w \in W. w \not\models \varphi$</td>
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<tr>
<td>in a frame</td>
<td>$\mathfrak{F}$ forces $\varphi$</td>
<td>$\mathfrak{F} \models \varphi$</td>
<td>$\forall \vartheta. \langle \mathfrak{F}, \vartheta \rangle \models \varphi$</td>
<td>$\exists \vartheta. \langle \mathfrak{F}, \vartheta \rangle \not\models \varphi$</td>
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Classicality: either $\bullet \models \varphi$ or else $\bullet \not\models \varphi$ holds for $\bullet \in \{w, \mathcal{M}, \mathfrak{F}\}$

Exercise: Work out the negation of each fully e.g. $\mathcal{M} \not\models \varphi$ is $\exists w \in W. w \models \neg \varphi$

Either $w \models \varphi$ or else $w \models \neg \varphi$ holds (Lemma 1)

But this does not apply to all: e.g. either $\mathcal{M} \models \varphi$ or else $\mathcal{M} \not\models \neg \varphi$ is rarely true.

$W \not\models \varphi$ meaning “every frame built out of given $W$ forces $\varphi$” is not interesting
Various Consequence Relations

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model.

Let $\mathcal{F}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame.

Let $\Gamma$ be a set of formulae, and $\varphi$ be a formula.

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Let $\bullet \models \Gamma$ stand for $\forall \psi \in \Gamma. \bullet \models \psi$ ($\bullet \in \{ w, \mathcal{M}, \mathfrak{F} \}$).

**World:** every world that forces $\Gamma$ also forces $\varphi$ \quad $\forall w \in W. w \models \Gamma \Rightarrow w \models \varphi$

**Model:** every model that forces $\Gamma$ also forces $\varphi$ \quad $\forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi$

**Frame:** every frame that forces $\Gamma$ also forces $\varphi$ \quad $\forall \mathfrak{F} \in \mathcal{F}. \mathfrak{F} \models \Gamma \Rightarrow \mathfrak{F} \models \varphi$

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Various Consequence Relations

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model.

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame.

Let $\Gamma$ be a set of formulae, and $\varphi$ be a formula.

We say

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Forces} & \text{We say} & \text{We write} & \text{When} & \bullet \models \neg \varphi \\
\hline
\text{in a world} & \text{w forces } \varphi & w \models \varphi & \vartheta(w, \varphi) = t & \vartheta(w, \varphi) = f \\
\hline
\text{in a model} & \text{\mathcal{M} forces } \varphi & \mathcal{M} \models \varphi & \forall w \in W. w \models \varphi & \exists w \in W. w \models \neg \varphi \\
\hline
\text{in a frame} & \text{\mathfrak{F} forces } \varphi & \mathfrak{F} \models \varphi & \forall \vartheta. \langle \mathfrak{F}, \vartheta \rangle \models \varphi & \exists \vartheta. \langle \mathfrak{F}, \vartheta \rangle \models \neg \varphi \\
\hline
\end{array}
\]

Let $\bullet \models \Gamma$ stand for $\forall \psi \in \Gamma. \bullet \models \psi \quad (\bullet \in \{w, \mathcal{M}, \mathfrak{F}\})$.

**World:** $\forall w \in W. w \models \Gamma \Rightarrow w \models \varphi$ iff $\forall w \in W. w \models \bigwedge \Gamma \rightarrow \varphi$ iff $\mathcal{M} \models \bigwedge \Gamma \rightarrow \varphi$

**Model:** $\forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi$ is the one we study

**Frame:** $\forall \mathfrak{F} \in \mathfrak{K}. \mathfrak{F} \models \Gamma \Rightarrow \mathfrak{F} \models \varphi$ usually undecidable
Logical Consequence, Validity and Satisfiability

**Logical Consequence:** \( \Gamma \models \phi \) iff \( \forall M \in \mathcal{K}. M \models \Gamma \Rightarrow M \models \phi \)

**Validity:** \( \phi \) is \( \mathcal{K} \)-valid iff \( \emptyset \models \phi \)

**Satisfiability:** \( \phi \) is \( \mathcal{K} \)-satisfiable iff \( \exists M = \langle W, R, \vartheta \rangle \in \mathcal{K}, \exists w \in W, w \models \phi \)

**Example:** \{\( p_0 \} \models []p_0 \). If every world in a model makes \( p_0 \) true, then every world in that model must make \( []p_0 \) true.

For a contradiction, assume \{\( p_0 \} \not\models []p_0 \).

i.e. exists \( M = \langle W, R, \vartheta \rangle \in \mathcal{K}. M \models p_0 \) and \( M \not\models []p_0 \).

i.e. exists \( w_0 \in W \) and \( w_0 \not\models []p_0 \)

i.e. exists \( w_0 \in W \) and \( w_1 \in W \) with \( w_0 R w_1 \) and \( w_1 \not\models p_0 \)

i.e. But \( M \models p_0 \) means \( \forall w \in W. w \models p_0 \), hence \( w_1 \models p_0 \) (contradiction)
Logical Consequence: Examples

**Example 1** All instances of $\varphi \rightarrow (\psi \rightarrow \varphi)$ are $\mathcal{K}$-valid.

For a contradiction, assume some instance $\varphi_1 \rightarrow (\psi_1 \rightarrow \varphi_1)$ not $\mathcal{K}$-valid.

i.e. exists model $\mathcal{M} = \langle W, R, \vartheta \rangle$ and $w \in W$ with $w \not\models \varphi_1 \rightarrow (\psi_1 \rightarrow \varphi_1)$.

i.e. $w \models \varphi_1$ and $w \not\models \psi_1 \rightarrow \varphi_1$.

i.e. $w \models \varphi_1$ and $w \models \psi_1$ and $w \not\models \varphi_1$. (contradiction)

**Exercise 1** All instances of $\neg\neg \varphi \rightarrow \varphi$ are $\mathcal{K}$-valid.

**Exercise 2** All instances of $(\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi))$ are $\mathcal{K}$-valid.
Logical Consequence: Examples

Example 2 All instances of $\square(\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$ are $\mathcal{K}$-valid.

For a contradiction, assume there is some instance $\square(\varphi_1 \rightarrow \psi_1) \rightarrow (\square \varphi_1 \rightarrow \square \psi_1)$ which is not $\mathcal{K}$-valid.

Therefore, there is some model $\mathcal{M} = \langle W, R, \vartheta \rangle$ and some $w \in W$ such that $w \not\models \square(\varphi_1 \rightarrow \psi_1) \rightarrow (\square \varphi_1 \rightarrow \square \psi_1)$.

i.e. $\vartheta(w, \square(\varphi_1 \rightarrow \psi_1) \rightarrow (\square \varphi_1 \rightarrow \square \psi_1)) = f$

i.e. $w \not\models \square(\varphi_1 \rightarrow \psi_1)$ and $w \not\models (\square \varphi_1 \rightarrow \square \psi_1)$

i.e. $w \models \square(\varphi_1 \rightarrow \psi_1)$ and $w \models \square \varphi_1$ and $w \not\models \square \psi_1$

i.e. $w \models \square(\varphi_1 \rightarrow \psi_1)$ and $w \models \square \varphi_1$ and $v \in W$ with $wRv$ and $v \not\models \psi_1$

i.e. $v \models \varphi_1 \rightarrow \psi_1$ and $v \models \varphi_1$ and $v \not\models \psi_1$

i.e. $v \models \psi_1$ and $v \not\models \psi_1$ (contradiction)
Logical Consequence: Examples

**Example 3** If $\phi \in \Gamma$ then $\Gamma \models \phi$  
(by definition of $\models$)

**Example 4** If $\Gamma \models \phi$ then $\Gamma \models \Box \phi$

For a contradiction, assume $\Gamma \models \phi$ and $\Gamma \not\models \Box \phi$.

i.e. exists $M = \langle W, R, \vartheta \rangle \not\models \Gamma$ and $w \in W$ with $w \models \neg \Box \phi$.

i.e. exists $M = \langle W, R, \vartheta \rangle \not\models \Gamma$ and $w \in W$ with $w \models \langle \rangle \neg \varphi$.

i.e. exists $M = \langle W, R, \vartheta \rangle \not\models \Gamma$ and $w \in W$ with $wRv$ and $v \models \neg \varphi$.

But $\Gamma \models \varphi$ means $\forall M \in \mathcal{K}. (M \not\models \Gamma \Rightarrow M \not\models \varphi)$, hence $v \not\models \varphi$. Contradiction.

**Exercise 3** If $\Gamma \models \varphi$ and $\Gamma \models \varphi \rightarrow \psi$ then $\Gamma \models \psi$
Logical Implication as Logical Consequence

Lemma 2  For any \( w \) in any model \( \langle W, R, \vartheta \rangle \), if \( w \models \{ \varphi, \varphi \rightarrow \psi \} \) then \( w \models \psi \)

Lemma 3  For any model \( M \), if \( M \models \{ \varphi, \varphi \rightarrow \psi \} \) then \( M \models \psi \)

Lemma 4  If \( \Gamma \models \varphi \rightarrow \psi \) then \( \Gamma, \varphi \models \psi \)  \hspace{1cm} (writing \( \Gamma, \varphi \) for \( \Gamma \cup \{ \varphi \} \))

Proof: Suppose \( \Gamma \models \varphi \rightarrow \psi \). Suppose \( M \models \Gamma, \varphi \). Must show \( M \models \psi \). But \( M \models \Gamma \) implies \( M \models \varphi \rightarrow \psi \), so \( M \models \{ \varphi, \varphi \rightarrow \psi \} \). Lemma 3 gives \( M \models \psi \).

Remark: Converse of Lemma 4 fails! e.g. We know \( p_0 \models []p_0 \). But \( \emptyset \models p_0 \rightarrow []p_0 \) is falsified in a model where \( w \models p_0 \) with \( wRv \) and \( v \models \neg p_0 \).

Lemma 5  If \( \Gamma, \varphi \models \psi \) then there exists an \( n \) such that

\[
\Gamma \models ([]^0 \varphi \land []^1 \varphi \land []^2 \varphi \land \cdots \land []^n \varphi) \rightarrow \psi
\]

where \( []^0 \varphi = \varphi \) and \( []^n \varphi = [][]^{n-1} \varphi \)  \hspace{1cm} (See Kracht for details)

e.g. \( p_0 \models []p_0 \) implies \( \emptyset \models (p_0 \land []p_0) \rightarrow []p_0 \) so \( n = 1 \) for this example
Summary: Logic = Syntax and Semantics

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \)  \((Atm)\)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \langle \rangle \varphi \mid [] \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \)  \((Fml)\)

Kripke Frame: directed graph \( \langle W, R \rangle \) where \( W \) is a non-empty set of points/worlds/vertices and \( R \subseteq W \times W \) is a binary relation over \( W \)

Valuation on a Kripke frame \( \langle W, R \rangle \) is a map \( \vartheta : W \times Atm \mapsto \{t, f\} \) telling us the truth value (t or f) of every atomic formula at every point in \( W \)

Kripke Model: \( \langle W, R, \vartheta \rangle \) where \( \vartheta \) is a valuation on a Kripke frame \( \langle W, R \rangle \)

Logical consequence: \( \Gamma \models \varphi \) iff \( \forall M \in K. M \models \Gamma \Rightarrow M \models \varphi \)

Having defined \( \Gamma \models \varphi \), we can consider a logic to be a set of formulae:
\[
K = \{\varphi \mid \emptyset \models \varphi\} = \{\varphi \mid \forall M \in K. M \models \varphi\} = \{\varphi \mid \forall \mathfrak{F} \in K. \mathfrak{F} \models \varphi\}
\]
**Lecture 2: Hilbert Calculi**

**Motivation:** Define a notion of deducibility “$\phi$ is deducible from $\Gamma$”

**Requirement:** Purely syntax manipulation, no semantic concepts allowed.

**Judgment:** $\Gamma \vdash \phi$ where $\Gamma$ is a finite set of assumptions (formulae)

Read $\Gamma \vdash \phi$ as “$\phi$ is derivable from assumptions $\Gamma$”

**Soundness:** If $\Gamma \vdash \phi$ then $\Gamma \models \phi$

If $\phi$ is derivable from $\Gamma$ then $\phi$ is a logical consequence of $\Gamma$

**Completeness:** If $\Gamma \models \phi$ then $\Gamma \vdash \phi$

If $\phi$ is a logical consequence of $\Gamma$ then $\phi$ is derivable from $\Gamma$

**Goal:** Deducibility captures logical consequence via syntax manipulation.
Hilbert Calculi: Derivation and Derivability

Assumptions: finite set of formulae accepted as derivable in one step (instantiation forbidden)

Axiom Schemata: Formula shapes, all of whose instances are accepted unquestionably as derivable in one step (listed shortly)

Rules of Inference: allow us to extend derivations into longer derivations

Judgment: $\Gamma \vdash \varphi$ where $\Gamma$ is a finite set of assumptions (formulae)

Rules: (Name) $\frac{\text{Judgment}_1 \ldots \text{Judgment}_n}{\text{Judgment}}$ (Condition) premisses conclusion

Read as: if premisses hold and condition holds then conclusion holds

Rule Instances: Uniformly replace formula variables and set variables in judgements with formulae and formula sets
Hilbert Derivability for Modal Logics

**Assumptions:** finite set of formulae accepted as derivable in one step  
(instantiation forbidden)

\[(\text{Id}) \quad \Gamma \vdash \varphi \quad \varphi \in \Gamma \quad \text{e.g. (Id)} \quad \{p_0\} \vdash p_0\]

**Axiom Schemata:** Formula shapes, all of whose instances are accepted  
unquestionably as derivable in one step  
(listed shortly)

\[(\text{Ax}) \quad \Gamma \vdash \varphi \quad \varphi \text{ is an instance of an axiom schema}\]

**Rules of Inference:** allow us to extend derivations into longer derivations

- **Modus Ponens** \((\text{MP})\) \[
\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}
\]

- **Necessitation** \((\text{Nec})\) \[
\frac{\Gamma \vdash \varphi}{\Gamma \vdash \Box \varphi}
\]
Hilbert Derivability for Modal Logics

(Id) $\Gamma \vdash \varphi \in \Gamma$

(Ax) $\Gamma \vdash \varphi$ is an instance of an axiom schema

(MP) $\Gamma \vdash \varphi , \Gamma \vdash \varphi \rightarrow \psi$

$\Gamma \vdash \psi$

(Nec) $\Gamma \vdash \varphi$

$\Gamma \vdash \square \varphi$

Rule Instances: Uniformly replace formula and set variables with formulae and formula sets

Derivation of $\varphi_0$ from assumptions $\Gamma_0$: is a finite tree of judgments with:

1. a root node $\Gamma_0 \vdash \varphi_0$

2. only (Ax) judgment instances and (Id) instances as leaves (sic!)

3. and such that all parent judgments are obtained from their child judgments by instantiating a rule of inference
Hilbert Calculus for Modal Logic $K$

Axiom Schemata:

**PC:** $\varphi \to (\psi \to \varphi)$
- $\neg\neg\varphi \to \varphi$
- $(\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi))$

**K:** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

**How used:** Create the leaves of a derivation via:

$$\begin{align*}
(Ax) & \quad \Gamma \vdash \varphi \quad \varphi \text{ is an instance of an axiom schema}
\end{align*}$$

- $\varphi \land \psi := \neg(\varphi \to \neg\psi)$
- $\varphi \lor \psi := (\neg\varphi \to \psi)$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \land (\psi \to \varphi)$
Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \rightarrow p_1\}$ and $\varphi_0 = [\square]p_1$. Usually omit braces.

Below is a derivation of $[\square]p_1$ from $\{p_0, p_0 \rightarrow p_1\}$.

$$
\frac{
\frac{p_0, p_0 \rightarrow p_1}{p_0} (Id) \\
\frac{p_0, p_0 \rightarrow p_1}{p_0 \rightarrow p_1} (Id)
}{p_0, p_0 \rightarrow p_1 \vdash p_1} (MP)
\frac{p_0, p_0 \rightarrow p_1 \vdash p_1}{p_0, p_0 \rightarrow p_1 \vdash [\square]p_1} (Nec)
$$

A derivation of $\varphi_0$ from assumptions $\Gamma_0$ is a finite tree of judgments with:

1. a root node $\Gamma_0 \vdash \varphi_0$
2. only (Ax) judgment instances and (Id) instances as leaves
3. and such that all parent judgments are obtained from their child judgments by instantiating a rule of inference
Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \rightarrow p_1\}$ and $\varphi_0 = [])p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \rightarrow p_1\}$.

\[
\begin{array}{c}
\begin{array}{c}
\text{(Id)} \quad p_0, p_0 \rightarrow p_1 \vdash p_0 \\
\text{(Id)} \quad p_0, p_0 \rightarrow p_1 \vdash p_0 \rightarrow p_1 \\
\text{(MP)} \quad p_0, p_0 \rightarrow p_1 \vdash p_1 \\
\text{(Nec)} \quad p_0, p_0 \rightarrow p_1 \vdash []p_1
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(Nec)} \quad \Gamma \vdash \varphi \\
\Gamma := \{p_0, p_0 \rightarrow p_1\} \\
\varphi := p_1
\end{array}
\]
Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \rightarrow p_1\}$ and $\varphi_0 = []p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \rightarrow p_1\}$.

$$
\begin{align*}
\Gamma_0 \vdash p_0, p_0 \rightarrow p_1 & \quad \text{(Id)} \\
\Gamma_0 \vdash p_0 & \quad \text{(Id)} \\
\Gamma_0 \vdash p_0 \rightarrow p_1 & \quad \text{(Id)} \\
\Gamma_0 \vdash p_0, p_0 \rightarrow p_1 \vdash p_0 & \quad \text{(MP)} \\
\Gamma_0 \vdash p_0, p_0 \rightarrow p_1 \vdash p_1 & \quad \text{(Nec)} \\
\Gamma_0 \vdash []p_1 & \\
\Gamma := \{p_0, p_0 \rightarrow p_1\} & \quad \varphi := p_0 & \quad \psi := p_1
\end{align*}
$$
Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \rightarrow p_1\}$ and $\varphi_0 = []p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \rightarrow p_1\}$.

\[
\begin{array}{c}
\Gamma_0 = \{p_0, p_0 \rightarrow p_1\} \\
\varphi_0 = [p_1] \\
\end{array}
\]

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Hilbert Derivations: Examples

Let $\Gamma = \{p_0, p_0 \rightarrow p_1\}$. Another derivation of $\Box p_1$ from $\{p_0, p_0 \rightarrow p_1\}$:

\[
\begin{align*}
\text{(Id)} & \quad p_0, p_0 \rightarrow p_1 \vdash p_0 \rightarrow p_1 \\
\text{(Nec)} & \quad p_0, p_0 \rightarrow p_1 \vdash \Box (p_0 \rightarrow p_1) \quad \text{(Ax)}
\end{align*}
\]

\[
\begin{align*}
\text{(MP)} & \quad p_0, p_0 \rightarrow p_1 \vdash \Box (p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1) \\
\end{align*}
\]

\[
\begin{align*}
\text{(Id)} & \quad p_0, p_0 \rightarrow p_1 \vdash p_0 \\
\text{(Nec)} & \quad p_0, p_0 \rightarrow p_1 \vdash \Box p_0 \\
\text{(Ax)} & \quad p_0, p_0 \rightarrow p_1 \vdash (\Box p_0) \rightarrow (\Box p_1) \quad \text{(MP)}
\end{align*}
\]

\[
\begin{align*}
\text{(Id)} & \quad p_0, p_0 \rightarrow p_1 \vdash p_0 \\
\text{(Nec)} & \quad p_0, p_0 \rightarrow p_1 \vdash \Box p_0 \\
\text{(Ax)} & \quad p_0, p_0 \rightarrow p_1 \vdash (\Box p_0) \rightarrow (\Box p_1) \quad \text{(MP)}
\end{align*}
\]

\[
\begin{align*}
\text{K:} & \quad \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \\
\end{align*}
\]

$\varphi := p_0$ \quad $\psi := p_1$
Summary: Logic = Syntax and Calculus

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \) \hspace{1cm} (Atm)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \langle \rangle \varphi \mid [] \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \) \hspace{1cm} (Fml)

Hilbert Calculus \( K \): \( [](\varphi \rightarrow \psi) \rightarrow ([]\varphi \rightarrow []\psi) \) \hspace{1cm} only modal axiom

(Id) \hspace{1cm} \frac{\Gamma \vdash \varphi}{\varphi \in \Gamma} \hspace{1cm} (Ax) \hspace{1cm} \frac{\Gamma \vdash \varphi}{\varphi \text{ is an instance of an axiom schema}}

(MP) \hspace{1cm} \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \rightarrow \psi}{\Gamma \vdash \psi}

(Nec) \hspace{1cm} \frac{\Gamma \vdash \varphi}{\Gamma \vdash []\varphi}

\( \Gamma \vdash \varphi \) iff there is a derivation of \( \varphi \) from \( \Gamma \) in \( K \).

Having defined \( \Gamma \vdash \varphi \), we can consider a logic to be a set of formulae:

\( K = \{ \varphi \mid \emptyset \vdash \varphi \} \)

\( \varphi \) is a theorem of \( K \) iff \( \varphi \in K \) \hspace{1cm} i.e. if it is deducible from the empty set

A modal logic is called “normal” if it extends \( K \) with extra modal axioms.
Soundness: all derivations are semantically correct

**Theorem:** if $\Gamma \vdash \psi$ then $\Gamma \models \psi$  
($\Gamma \models \psi$ means $\forall M \in K. M \models \Gamma \Rightarrow M \models \psi$)

**Proof:** By induction on the length $l$ of the derivation of $\Gamma \vdash \psi$

$l = 0$: So $\Gamma \vdash \psi$ because $\psi \in \Gamma$. But $M \models \Gamma$ implies $M \models \psi$ for all $\psi \in \Gamma$.

$l = 0$: So $\Gamma \vdash \psi$ because $\psi$ is an axiom schema instance. By Eg 1, Ex 1, Ex 2, Eg 2, we know $\emptyset \models \psi$ for every axiom schema instance $\psi$, hence $\Gamma \models \psi$.

**Ind. Hyp.:** Theorem holds for all derivations of length less than some $k > 0$.

**Ind. Step:** Suppose $\Gamma \vdash \psi$ has a derivation of length $k$. Bottom-most rule?

**MP:** So both $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ are shorter than $k$. By IH $\Gamma \models \varphi \rightarrow \psi$ and $\Gamma \models \varphi$. But if $w \models \varphi \rightarrow \psi$ and $w \models \varphi$ then $w \models \psi$, hence $\Gamma \models \psi$.

**Nec:** Then we know that $\Gamma \vdash \psi$ has length shorter than $k$. By IH we know $\Gamma \models \psi$. But if $\Gamma \models \psi$ then $\Gamma \models []\psi$ by Eg 4.
Completeness: all semantic consequences are derivable

Theorem: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

Proof Method: Prove contrapositive, if $\Gamma \not\models \varphi$ then $\Gamma \not\models \varphi$

Proof Plan: Assume $\Gamma \not\models \varphi$. Show there is a $\mathcal{K}$-model $\mathcal{M}_c = \langle W_c, R_c, \vartheta_c \rangle$

such that $\mathcal{M}_c \not\models \Gamma$ and $\mathcal{M}_c \not\models \varphi$ (i.e. $\exists w \in W_c. w \models \neg \varphi$)

Technique: is known as the canonical model construction

Local Consequence: Write $X \vdash_l \varphi$ iff there exists a finite subset

$\{\psi_1, \psi_2, \ldots, \psi_n\} \subseteq X$ such that $\emptyset \vdash (\psi_1 \land \psi_2 \land \cdots \land \psi_n) \rightarrow \varphi$

Exercise: if $X \vdash_l \varphi$ then $X \vdash \varphi$ by (MP) on $X \vdash \land (\psi_i)$ and $X \vdash \land (\psi_i) \rightarrow \varphi$

Set $X$ is Maximal: if $\forall \psi. \psi \in X$ or $\neg \psi \in X$

Set $X$ is Consistent: if both $X \vdash_l \psi$ and $X \vdash_l \neg \psi$ never hold, for any $\psi$

Set $X$ is Maximal-Consistent: if it is maximal and consistent.
Lindenbaum’s Construction of Maximal-Consistent Sets

Lemma 6  Every consistent $\Gamma$ is extendable into a maximal-consistent $X^* \supset \Gamma$.

Proof:  Choose an enumeration $\varphi_1, \varphi_2, \varphi_3, \cdots$ of the set of all formulae.

Stage 0:  Let $X_0 := \Gamma$

Stage $n > 0$:  $X_n := \begin{cases} X_{n-1} \cup \{\varphi_n\} & \text{if } X_{n-1} \vdash \varphi_n \\ X_{n-1} \cup \{\neg \varphi_n\} & \text{otherwise} \end{cases}$

Stage $\omega$:  $X^* := \bigcup_{n=0}^{\omega} X_n$

Question: Every Stage is deterministic so why is $X^*$ not unique?  (choice)

Not Effective:  Relies on classicality: either $X_{n-1} \vdash \varphi_n$ or $X_{n-1} \not\vdash \varphi_n$ is true, but does not say how we decide the question.

Exercise:  Why is having both $X_{n-1} \vdash \varphi_n$ and $X_{n-1} \vdash \neg \varphi_n$ impossible?
Lindenbaum’s Construction of Maximal-Consistent Sets

**Lemma 7** *Every consistent* $\Gamma$ *is extendable into a maximal-consistent* $X^* \supset \Gamma$.

**Proof:** Choose an enumeration $\varphi_1, \varphi_2, \varphi_3, \cdots$ of the set of all formulae.

**Stage 0:** Let $X_0 := \Gamma$

**Stage $n > 0$:** $X_n := \begin{cases} X_{n-1} \cup \{\varphi_n\} & \text{if } X_{n-1} \vdash \varphi_n \\ X_{n-1} \cup \{\neg \varphi_n\} & \text{otherwise} \end{cases}$

**Stage $\omega$:** $X^* := \bigcup_{n=0}^{\omega} X_n$

**Chain of consistent sets:** $X_0 \subset X_1 \subset \cdots$

**Maximality:** Clearly, for all $\varphi$ either $\varphi \in X^*$ or else $\neg \varphi \in X^*$

**$X^*$ is consistent:** Suppose for a contradiction that $X^*$ is inconsistent. Thus $X^* \vdash \psi$ and $X^* \vdash \neg \psi$ for some $\psi$. Hence $\psi \in X_i$ and $\neg \psi \in X_j$ for some $i$ and $j$. Let $k := \max\{i, j\}$. Then $X_k \vdash \psi$ since $X_i \subseteq X_k$ and $X_k \vdash \neg \psi$ since $X_j \subseteq X_k$. Contradiction since $X_k$ is consistent.
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* \mid X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w R_c v \iff \{ \varphi \mid []\varphi \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} t & \text{if } p \in w \\ f & \text{otherwise} \end{cases}$

Claim: $w R_c v \iff \{ \langle \rangle \varphi \mid \varphi \in v \} \subseteq w$

Proof left to right: Suppose $w R_c v$ and $\{ \langle \rangle \varphi \mid \varphi \in v \} \not\subseteq w$. Hence, there is some $\varphi \in v$ such that $\langle \rangle \varphi \not\in w$. By maximality, $\neg \langle \rangle \varphi \in w$. By consistency, $[]\neg \varphi \in w$. By definition of $w R_c v$, we must have $\neg \varphi \in v$. Contradiction.

Proof right to left: Suppose $\{ \langle \rangle \varphi \mid \varphi \in v \} \subseteq w$ and not $w R_c v$. Hence, there is some $[]\varphi \in w$ such that $\varphi \not\in v$. By maximality, $\neg \varphi \in v$. By supposition, $\langle \rangle \neg \varphi \in w$. By consistency, $\neg []\varphi \in w$. Contradiction.
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* \mid X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w \mathrel{R_c} v \iff \{ \varphi \mid []\varphi \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} 
  t & \text{if } p \in w \\
  f & \text{otherwise}
\end{cases}$

**Lemma 8** For every formula $\varphi$ and every formula $\psi$ and every $w \in W_c$:

- $\neg:\quad \neg\varphi \in w \iff \varphi \not\in w$  
  i.e. $\neg\varphi \not\in w$ iff $\varphi \in w$

- $\land:\quad \varphi \land \psi \in w \iff \varphi \in w \text{ and } \psi \in w$

- $\lor:\quad \varphi \lor \psi \in w \iff \varphi \in w \text{ or } \psi \in w$

- $\to:\quad \varphi \to \psi \in w \iff \varphi \not\in w \text{ or } \psi \in w$

- $[]:\quad []\varphi \in w \iff \forall v \in W_c. w R_c v \Rightarrow \varphi \in v$

- $\langle \rangle:\quad \langle \rangle\varphi \in w \iff \exists v \in W_c. w R_c v \& \varphi \in v$
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* \mid X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w R_c v$ iff $\{ \varphi \mid \Box \varphi \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} 
t & \text{if } p \in w \\ 
f & \text{otherwise} \end{cases}$

Claim: $\varphi \land \psi \in w$ iff $\varphi \in w$ and $\psi \in w$

Proof right to left: Suppose $\varphi \land \psi \in w$ and $\varphi \not\in w$. Then $\neg \varphi \in w$.

Note $(\varphi \land \psi) \rightarrow \varphi \in w$ since $\emptyset \vdash (\varphi \land \psi) \rightarrow \varphi$ by PC (exercise)

Exists $k$ with $X_k \vdash \neg \varphi$, and $X_k \vdash \varphi \land \psi$, and $X_k \vdash (\varphi \land \psi) \rightarrow \varphi$, by (Id).

Then $X_k \vdash \varphi$ by (MP) Contradiction.

Proof left to right: Suppose $\varphi \in w$ and $\psi \in w$ and $\varphi \land \psi \not\in w$.

i.e. $(\varphi \rightarrow \neg \psi) \in w$ since $\varphi \land \psi := \neg (\varphi \rightarrow \neg \psi)$

i.e. exists $k$ such that $X_k \vdash \varphi$ and $X_k \vdash \varphi \rightarrow \neg \psi$ and $X_k \vdash \psi$ by (id)

Then $X_k \vdash \neg \psi$ by (MP) Contradiction
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* \mid X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w R_c v \iff \{ \psi \mid \square \psi \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} 
  t & \text{if } p \in w \\
  f & \text{otherwise}
\end{cases}$

Claim: $\square \varphi \in w \iff \forall v \in W_c. (w R_c v \Rightarrow \varphi \in v)$

Proof left to right: Suppose $\square \varphi \in w$ and not $\forall v \in W_c. w R_c v \Rightarrow \varphi \in v$

i.e. $\square \varphi \in w$ and $\exists v \in W_c. w R_c v \& \varphi \notin v$

i.e. $\square \varphi \in w$ and $\exists v \in W_c. \varphi \in v \& \varphi \notin v$

Contradiction.
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* \mid X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w R_c v$ iff $\{ \psi \mid \Box \psi \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} 
    t & \text{if } p \in w \\
    f & \text{otherwise}
\end{cases}$

Claim: $\Box \varphi \in w$ iff $\forall v \in W_c. (w R_c v \Rightarrow \varphi \in v)$

Proof right to left: Suppose $\forall v \in W_c. (w R_c v \Rightarrow \varphi \in v)$. Must show $\Box \varphi \in w$.

i.e. $\forall v \in W_c. (\{ \psi \mid \Box \psi \in w \} \subseteq v \Rightarrow \varphi \in v)$

Let $\Psi := \bigwedge \{ \psi \mid \Box \psi \in w \}$

i.e. $\forall v \in W_c. (\Psi \in v \Rightarrow \varphi \in v)$

i.e. $\forall v \in W_c. \Psi \rightarrow \varphi \in v$ by Lemma 8($\rightarrow$).

i.e. $\Gamma \vdash \Psi \rightarrow \varphi$ (else can choose $\varphi_0 = \Psi \rightarrow \varphi$ for some $v$)

i.e. $\Gamma \vdash \Box (\Psi \rightarrow \varphi)$ by (Nec)

Note $\Gamma \vdash \Box (\Psi \rightarrow \varphi) \rightarrow (\Box \Psi \rightarrow \Box \varphi)$ by (Ax)

Hence $\Gamma \vdash \Box (\Box \Psi \rightarrow \Box \varphi)$ by (MP)

Hence $\Box (\Box \Psi \rightarrow \Box \varphi) \in w$.

Note, $\emptyset \vdash ((\Box \psi_0) \land (\Box \psi_1)) \rightarrow (\Box (\psi_0 \land \psi_1))$

Hence $\{ \Box \Psi, (\Box \Psi \rightarrow \Box \varphi) \} \subseteq w$.

Hence $\Box \varphi \in w$ by (MP).
Truth Lemma

Lemma 9  For every $\varphi$ and every $w \in W_c$: $\vartheta_c(w, \varphi) = t$ iff $\varphi \in w$.

Proof: Pick any $\varphi$, any $w \in W$. Proceed by induction on length $l$ of $\varphi$.

$l = 0$: So $\varphi = p$ is atomic. Then, $\vartheta_c(w, p) = t$ iff $p \in w$ by definition of $\vartheta_c$.

Ind. Hyp.: Lemma holds for all formulae with length $l$ less than some $n > 0$

Ind. Step: Assume $l = n$ and proceed by cases on main connective

$\varphi = [\psi]$: We have $\vartheta_c(w, [\psi]) = t$

iff $\forall v \in W_c. (w R_c v \Rightarrow \vartheta_c(v, \psi) = t)$ (by defn of valuations $\vartheta$)

iff $\forall v \in W_c. (w R_c v \Rightarrow \psi \in v)$ (by IH)

iff $[\psi] \in w$ by Lemma 8([]).

Exercise: complete the proof
Completeness Proof

**Corollary 1**  \( \langle W_c, R_c, \vartheta_c \rangle \models \Gamma \)

**Proof:** Since \( \Gamma \) is in every maximal-consistent set extending it, we must have \( \Gamma \subset w \) for all \( w \in W_c \). By Lemma 9, \( w \models \Gamma \), hence \( \langle W_c, R_c, \vartheta_c \rangle \models \Gamma \)

**Proof of Completeness:** if \( \Gamma \not\models \varphi \) then \( \Gamma \not\models \varphi \)

Suppose \( \Gamma \not\models \varphi \). Hence \( \Gamma \not\models \perp \varphi \). Construct the canonical model \( \mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle \). Consider any ordering of formulae where \( \varphi \) is the first formula and let the associated maximal-consistent extension of \( \Gamma \) be \( X^* \).

Since \( \Gamma \not\models \perp \varphi \) we must have \( \neg \varphi \in X^* \). The set \( X^* \) appears as some world \( w_0 \in W_c \) (say). Hence there exists at least one world where \( \neg \varphi \in w_0 \). By Lemma 9 \( w_0 \models \neg \varphi \) i.e. \( \mathcal{M}_\Gamma \models \neg \varphi \). By Corollary 1, we know \( \mathcal{M}_\Gamma \models \Gamma \). Since the canonical model is a Kripke model, we have \( \Gamma \not\models \varphi \). (i.e. not \( \forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi \))

**Completeness:** By contraposition, if \( \Gamma \models \varphi \) then \( \Gamma \models \varphi \).
Notes

\( \Gamma \vdash \varphi \) iff \( \Gamma \models \varphi \) relies on the canonical frame \( \langle W_c, R_c \rangle \) being a Kripke frame by its definition. (i.e. \( \langle W_c, R_c \rangle \in \mathcal{K} \))

Later we shall see that the canonical model is not always sound for \( \vdash \): that is we can have \( \varphi \) where \( \Gamma \vdash \varphi \) and \( M_\Gamma \not\models \varphi \) (incomplete logics)

Beware: some books (e.g. Goldblatt) use the notation \( \Gamma \vdash \varphi \) for our \( \Gamma \vdash l \varphi \) because then the deduction theorem holds: \( \Gamma, \varphi \vdash l \psi \) iff \( \Gamma \vdash l \varphi \to \psi \)

Exercise: Prove it.

For us, the syntactic counterparts of Lemma 4 and Lemma 5 are:

**Lemma 10** \( \Gamma \vdash \varphi \to \psi \) implies \( \Gamma, \varphi \vdash \psi \)

**Lemma 11** \( \Gamma, \varphi \vdash \psi \) implies \( \exists n. \Gamma \vdash \boxdot^0 \varphi \land \cdots \land \boxdot^n \varphi \to \psi \)
Lecture 3: Logic = Syntax and (Semantics or Calculus)

$\Gamma \models \varphi$ : semantic consequence in class of Kripke models $\mathcal{K}$

$\Gamma \vdash \varphi$ : deducibility in Hilbert calculus $\mathcal{K}$

Soundness: if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness: if $\Gamma \not\vdash \varphi$ then $\mathcal{M}_\Gamma \not\models \varphi$ and $\mathcal{M}_\Gamma \in \mathcal{K}$.

$$\mathcal{K} = \{ \varphi \mid \emptyset \models \varphi \}$$ the validities of Kripke frames $\mathcal{R}$

$$\mathcal{K} = \{ \varphi \mid \emptyset \vdash \varphi \}$$ the theorems of Hilbert calculus $\mathcal{K}$

**Theorem 1** $\mathcal{K} = \mathcal{K}$

The presence of $R$ makes modal logics non-truth-functional.

But Kripke models put no conditions on $R$.

So what happens if we put conditions on $R$?
Valid Shapes and Frame Conditions

A binary relation $R$ is reflexive if $\forall w \in W. wRw$.

A frame $\langle W, R \rangle$ or model $\langle W, R, \vartheta \rangle$ is reflexive if $R$ is reflexive.

The shape $[]\varphi \rightarrow \varphi$ is called $T$.

A frame $\langle W, R \rangle$ validates a shape iff it forces all instances of that shape.

i.e. for all instances $\psi$ of the shape and all valuations $\vartheta$ we have $\langle W, R, \vartheta \rangle \models \psi$

Lemma 12 A frame $\langle W, R \rangle$ validates $T$ iff $R$ is reflexive.

Intuition: the shape $T$ captures or corresponds to reflexivity of $R$. 

Valid Shapes and Frame Conditions

A relation \( R \) is reflexive if \( \forall w \in W. wRw \). The shape \( []\varphi \rightarrow \varphi \) is called \( T \).

**Lemma 13** [Correspondence] A frame \( \langle W, R \rangle \) validates \( T \) iff \( R \) is reflexive.

**Proof(i):** Assume \( R \) is reflexive and \( \langle W, R \rangle \not \models []\psi \rightarrow \psi \) for some \( []\psi \rightarrow \psi \).

Exists model \( \langle W, R, \vartheta \rangle \) and \( w_0 \in W \) with \( w_0 \not \models []\psi \) and \( w_0 \not \models \psi \).

\( v \models \psi \) for all \( v \) with \( w_0Rv \) \( \quad w_0Rw_0 \) \quad \text{Hence,} \quad w_0 \not \models \psi \). Contradiction

**Proof(ii):** Assume \( \langle W, R \rangle \) forces all instances of \( []\varphi \rightarrow \varphi \), and \( R \) not reflexive.

Exists \( w_0 \in W \) such that \( w_0Rw_0 \) does not hold.

For all \( w \in W \), let \( \vartheta(w, p_0) = t \) iff \( w_0Rw \). \quad \text{(we define} \vartheta)\)

\( \vartheta(v, p_0) = t \) for every \( v \) with \( w_0Rv \), and \( \vartheta(w_0, p_0) = f \) since not \( w_0Rw_0 \).

\( w_0 \models []p_0 \) and \( w_0 \not \models p_0 \) hence \( w_0 \not \models []p_0 \rightarrow p_0 \)

But \( []p_0 \rightarrow p_0 \) is an instance of \( T \) hence \( w_0 \models []p_0 \rightarrow p_0 \). Contradiction.
Valid Shapes and Frame Conditions

A frame $\langle W, R \rangle$ is reflexive if $\forall w \in W. wRw$. The shape $\Box \varphi \rightarrow \varphi$ is called $T$.

A frame $\langle W, R \rangle$ validates $T$ iff $R$ is reflexive.

This correspondence does not work for models!

A model $\langle W, R, \vartheta \rangle$ validates $T$ iff $R$ is reflexive is false!

Consider the reflexive model $M$ where:
$W = \{w_0\}$ and $R = \{(w_0, w_0)\}$ and $\vartheta$ is arbitrary.

This model must validate $T$ since $\langle W, R \rangle$ is reflexive.

Now consider the model $M'$ where:
$W' = \{v_0, v_1\}$
$R' = \{(v_0, v_1), (v_1, v_0)\}$
$\vartheta'$ is:

$$
\vartheta'(v_i, p) = \begin{cases} 
  t & \text{if } \vartheta(w_0, p) = t \\
  f & \text{otherwise}
\end{cases}
$$

Exercise: model $M'$ also validates $T$. But $M'$ is not reflexive!
Summary: The Logic of Reflexive Kripke Frames

Let $\mathcal{K} \mathcal{I}$ be the class of all reflexive Kripke frames.

Let $\mathcal{K} \mathcal{T}$ be the class of all reflexive Kripke models.

Let $\mathcal{K} \mathcal{T} = \mathcal{K} + [] \varphi \rightarrow \varphi$ (shape $T$) as an extra modal axiom.

Define $\Gamma \models_{\mathcal{K} \mathcal{T}} \varphi$ to mean $\forall M \in \mathcal{K} \mathcal{T}. M \Vdash \Gamma \Rightarrow M \Vdash \varphi$.

Define $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ to mean there is a derivation of $\varphi$ from $\Gamma$ in $\mathcal{K} \mathcal{T}$.

Soundness: if $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ then $\Gamma \models_{\mathcal{K} \mathcal{T}} \varphi$

Proof: all instances of $T$ are valid in reflexive frames.

Completeness: if $\Gamma \not\vdash_{\mathcal{K} \mathcal{T}} \varphi$ then $M_\Gamma \not\models_{\mathcal{K} \mathcal{T}} \varphi$ and $M_\Gamma \in \mathcal{K} \mathcal{T}$

Proof: if $M_\Gamma$ validates (all instances of) $T$ then $M_\Gamma$ is reflexive. (sic!)

i.e. $T$-instance $[] \psi_1 \rightarrow \psi_1 \in w$ iff $[] \psi_1 \in w \Rightarrow \psi_1 \in w$ by Lemma 8($\rightarrow$).

$\forall w, v \in W. w R_c v$ iff $\{ \psi \mid [] \psi \in w \} \subseteq v$ implies $w R_c w$
### More Axiom and Frame Correspondences

<table>
<thead>
<tr>
<th>Name</th>
<th>Axiom</th>
<th>Frame Class</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$[] \varphi \to \varphi$</td>
<td>Reflexive</td>
<td>$\forall w \in W. wRw$</td>
</tr>
<tr>
<td>D</td>
<td>$[[]] \varphi \to [\langle \rangle] \varphi$</td>
<td>Serial</td>
<td>$\forall w \in W. \exists v \in W. wRv$</td>
</tr>
<tr>
<td>4</td>
<td>$[\langle \rangle] \varphi \to [\langle \rangle] [\langle \rangle] \varphi$</td>
<td>Transitive</td>
<td>$\forall u, v, w \in W. uRv &amp; vRw \Rightarrow uRw$</td>
</tr>
<tr>
<td>5</td>
<td>$[\langle \rangle] [\langle \rangle] \varphi \to [\langle \rangle] \varphi$</td>
<td>Euclidean</td>
<td>$\forall u, v, w \in W. uRv &amp; uRw \Rightarrow vRw$</td>
</tr>
<tr>
<td>B</td>
<td>$\varphi \to [\langle \rangle] \varphi$</td>
<td>Symmetric</td>
<td>$\forall u, v \in W. uRv \Rightarrow vRu$</td>
</tr>
<tr>
<td>Alt₁</td>
<td>$[\langle \rangle] \varphi \to [\langle \rangle] [\langle \rangle] \varphi$</td>
<td>Weakly-Functional</td>
<td>$\forall u, v, w \in W. uRv &amp; uRw \Rightarrow v = w$</td>
</tr>
<tr>
<td>2</td>
<td>$[\langle \rangle] [\langle \rangle] \varphi \to [\langle \rangle] [\langle \rangle] \varphi$</td>
<td>Weakly-Directed</td>
<td>$\forall u, v, w \in W. uRv &amp; uRw \Rightarrow \exists x \in W. vRx &amp; wRx$</td>
</tr>
<tr>
<td>3</td>
<td>$[\langle \rangle] \varphi \land [\langle \rangle] \psi \to [\langle \rangle] (\varphi \land [\langle \rangle] \psi)$</td>
<td>Weakly-Linear</td>
<td>$\forall u, v, w \in W. uRv &amp; uRw \Rightarrow vRw \lor wRv \lor w = v$</td>
</tr>
</tbody>
</table>

Let $KA_1A_2 \cdots A_n = K + A_1 + A_2 + \cdots + A_n$. (any $A_i$s from above)

**Theorem 2** $\Gamma \vdash KA_1A_2\cdots A_n \varphi$ iff $\Gamma \models KA_1A_2\cdots A_n \varphi$
Correspondence, Canonicity and Completeness

Normal modal logic $L$ is determined by class of Kripke frames $\mathcal{C}$ if:
$$\forall \varphi. \mathcal{C} \models \varphi \iff \vdash_L \varphi.$$ Normal modal logic $L$ is complete if determined by some class of Kripke frames. A normal modal logic is canonical if it is determined by its canonical frame.

A Sahlqvist formula is a formula with a particular shape (too complicated to define here but see Blackburn, de Rijke and Venema)

**Theorem 3** Every Sahlqvist formula $\varphi$ corresponds to some first-order condition on frames, which is effectively computable from $\varphi$.

**Theorem 4** If each axiom $A_i$ is a Sahlqvist formula, then the Hilbert logic $K A_1 A_2 \cdots A_n$ is canonical, and is determined by a class of frames which is first-order definable.

**Theorem 5** Given a collection of Sahlqvist axioms $A_1, \cdots, A_k$, the logic $K A_1 A_2 \cdots A_k$ is complete wrt the class of frames determined by $A_1 \cdots A_k$.  

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Not All First-Order Conditions Are Captured By Shapes

**Theorem 6 (Chagrov)**  It is undecidable whether an arbitrary modal formula has a first-order correspondent.

Question: Are there conditions on $R$ not captured by any shape?

Yes: the following conditions cannot be captured by any shape:

**Irreflexivity:** $\forall w \in W. \text{not } wRw$

**Anti-Symmetry:** $\forall u, v \in W. uRv \& vRu \Rightarrow u = v$

**Asymmetry:** $\forall u, v \in W. uRv \Rightarrow \text{not } (vRu)$

See Goldblatt for details.
Second-Order Aspects of Modal Logics

All of these conditions are first-order definable so it looked like modal logic was just a fragment of first-order logic ...

An $R$-chain is a sequence of distinct worlds $w_0 Rw_1 Rw_2 \cdots$.

<table>
<thead>
<tr>
<th>Name</th>
<th>Shape</th>
<th>$R$ Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$[]([]\varphi \to \varphi) \to []\varphi$</td>
<td>transitive and no infinite $R$-chains</td>
</tr>
<tr>
<td>$Grz$</td>
<td>$[]([](\varphi \to []\varphi) \to \varphi) \to []\varphi$</td>
<td>reflexive, transitive and no infinite $R$-chains</td>
</tr>
</tbody>
</table>

The condition “no infinite $R$-chains” is not first-order definable since “finiteness” is not first-order definable. It requires second-order logic, so propositional modal logic is a fragment of quantified second-order logic.

The logic $KG$ has an interesting interpretation where $[]\varphi$ can be read as “$\varphi$ is provable in Peano Arithmetic”.

These logics are not Sahlqvist.
Consider logic $\mathbf{KH}$ where $H$ is the axiom schema $\Box(\Box \varphi \leftrightarrow \varphi) \rightarrow \Box \varphi$.

**Theorem 7 (Boolos and Sambin)** *The logic $\mathbf{KH}$ is not determined by any class of Kripke frames.*


Incompleteness first found in modal logic by S K Thomason in 1972. Beware, there is also a R H Thomason in modal logic literature.

Can regain a general frame correspondence by using *general frames* instead of Kripke frames: see Kracht.

Kracht shows how to compute modal Sahlqvist formulae from first-order formulae.

SCAN Algorithm of Dov Gabbay and Hans Juergen Ohlbach automatically computes first-order equivalents via the web.
Sub-Normal Mono-Modal Logics

Hilbert Calculus $S = PC$ plus modal axioms (not $K$)

(Id) \[ \Gamma \vdash_s \varphi \quad \varphi \in \Gamma \]

(Ax) \[ \Gamma \vdash_s \varphi \quad \varphi \text{ is an instance of an axiom schema} \]

(MP) \[ \Gamma \vdash_s \varphi, \Gamma \vdash_s \varphi \rightarrow \psi \quad \frac{\Gamma \vdash_s \varphi \rightarrow \psi}{\Gamma \vdash_s \psi} \]

(Mon) \[ \Gamma \vdash_s \varphi \rightarrow \psi \quad \frac{\Gamma \vdash_s [] \varphi \rightarrow [] \psi}{\text{no rule (Nec)}} \]

$\Gamma \vdash_s \varphi$ : iff there is a derivation of $\varphi$ from $\Gamma$ in $S$.

Such modal logics are called “sub-normal”.

$\Gamma \models_s \varphi$: needs Kripke models $\langle W, Q, R, \vartheta \rangle$ where: $W$ is a set of “normal” worlds and $\vartheta$ behaves as usual, and $Q$ is a set of “queer” or “non-normal” worlds where $\vartheta(w_q, \langle \varphi \rangle) = t$ for all $\varphi$ and all $w_q \in Q$ by definition. Then (Nec) fails since $\mathcal{M} \models \varphi \nrightarrow \mathcal{M} \models [] \varphi$ i.e. every non-normal world makes $[] \varphi$ false.

Applications in logics for agents: $\models \varphi \Rightarrow \models [] \varphi$ says that “if $\varphi$ is valid, then $\varphi$ is known”, but agents may not be omniscient, hence want to go “sub-normal”.

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Regaining Expressive Power Via Nominals

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \)  \((Atm)\)

Nominals: \( i ::= i_0 \mid i_1 \mid i_2 \mid \cdots \) \((Nom)\)

Formulae: \( \varphi ::= p \mid i \mid \neg \varphi \mid \langle \varphi \rangle \mid [] \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \) \((Fml)\)

Valuation: for every \( i \), \( \vartheta(w, i) = t \) at only one world

Intuition: \( i \) is the name of \( w \)

Expressive Power:

**Irreflexivity:** \( \forall w \in W. \neg wRw \) \( i \rightarrow \neg \langle \rangle i \)

**Anti-Symmetry:** \( \forall u, v \in W. uRv \land vRu \Rightarrow u = v \) \( i \rightarrow [](\langle \rangle i \rightarrow i) \)

**Asymmetry:** \( \forall u, v \in W. uRv \Rightarrow \neg (vRu) \) \( i \rightarrow \neg \langle \rangle \langle \rangle i \)

Motivation: Finding derivations in Hilbert Calculi is cumbersome:

\[ \Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \varphi \rightarrow \psi \text{ fails!} \]

\[ \Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash (\Box^0 \varphi \land \Box^1 \varphi \land \cdots \land \Box^n \varphi) \rightarrow \psi \]

\[ \vdash \xi \]
\[ \vdash \xi \rightarrow (\varphi \rightarrow \psi) \]
(\text{MP})
\[ \vdash \xi \]
\[ \vdash \xi \rightarrow (\varphi \rightarrow \psi) \]
(\text{MP})
\[ \vdash \varphi \rightarrow \psi \]

Resolution: one rule suffices for classical first-order logic, but not so for modal resolution

Decidability: questions can be answered via refinements of canonical models called filtrations, but there are better ways ...

For filtrations see Goldblatt.
Negation Normal Form

NNF: A formula is in negation normal form iff all occurrences of ¬ appear in front of atomic formulae only, and there are no occurrences of →.

Lemma 14 Every formula \( \varphi \) can be rewritten into a formula \( \varphi' \) such that \( \varphi' \) is in negation normal form, the length of \( \varphi' \) is at most polynomially longer than the length of \( \varphi \), and \( \emptyset \models \varphi \leftrightarrow \varphi' \).

Proof: Repeatedly distribute negation over subformulae using the following valid principles:

\[
\models (\varphi_1 \to \psi_1) \leftrightarrow (\neg \varphi_1 \lor \psi_1) \quad \models (\neg \varphi_1 \lor \psi_1) \leftrightarrow (\varphi_1 \land \neg \psi_1)
\]

\[
\models (\neg \varphi \land \psi) \leftrightarrow (\neg \varphi \lor \neg \psi) \quad \models (\neg \varphi \lor \psi) \leftrightarrow (\neg \varphi \land \neg \psi) \quad \models \neg \neg \varphi \leftrightarrow \varphi
\]

\[
\models \neg \langle \rangle \varphi \leftrightarrow [\neg \varphi
\]

Beware: if \( \leftrightarrow \) is a primitive connective then this blows up!
Examples: NNF

Example:
\[
\neg (\Box (p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1)) \\
(\Box (p_0 \rightarrow p_1) \land \neg (\Box p_0 \rightarrow \Box p_1)) \\
(\Box (p_0 \rightarrow p_1) \land (\Box p_0 \land \neg \Box p_1)) \\
(\Box (\neg p_0 \lor p_1) \land (\Box p_0 \land \langle \rangle \neg p_1))
\]

Example:
\[
\neg (\Box p_0 \rightarrow p_0) \\
(\Box p_0) \land (\neg p_0) \\
\neg (\Box p_0 \rightarrow \Box \Box p_0) \\
(\Box p_0) \land (\neg \Box \Box p_0) \\
(\Box p_0) \land (\langle \rangle \neg \Box p_0) \\
(\Box p_0) \land (\langle \rangle \langle \rangle \neg p_0)
\]
Tableau Calculi for Normal Modal Logics

Static Rules: (id) \[ \frac{p; \neg p; X}{\times} \]

(\&) \[ \frac{\varphi \land \psi; X}{\varphi; \psi; X} \]

(\|) \[ \frac{\varphi \lor \psi; X}{\varphi; X \mid \psi; X} \]

Transitional Rule: \(\langle\rangle K\) \[ \frac{\langle\rangle \varphi; [] X; Z}{\varphi; X} \]

\[ \forall \psi. [] \psi \not\in Z \]

\[ [] X = \{ [] \psi \mid \psi \in X \} \]

\(X, Y, Z\) are possibly empty multisets of formulae and \(\varphi; X\) stands for \(\{ \varphi \}\) multiset-union \(X\) so number of occurrences matter

Rules: (Name) \[ \frac{\text{MSet}}{\text{MSet}_1 \mid \ldots \mid \text{MSet}_n} \]

if numerator is \(\mathcal{K}\)-satisfiable then some denominator is \(\mathcal{K}\)-satisfiable

A \textbf{K-tableau} for \(Y\) is an inverted tree of nodes with:

1. a root node \text{nff} \(Y\)

2. and such that all children nodes are obtained from their parent node by instantiating a rule of inference

A \textbf{K-tableau} is closed (derivation) if all leaves are (id) instances, else it is open.
Examples of $\mathbf{K}$-Tableau

\[
\begin{align*}
\text{(id)} & \quad p; \neg p; X \quad (\wedge) \quad \varphi \wedge \psi; X \quad (\vee) \quad \varphi \vee \psi; X \\
& \quad \langle \rangle \quad \langle \rangle \quad \langle \rangle \quad \langle \rangle
\end{align*}
\]

\[
\begin{align*}
\neg(\langle \rangle(p_0 \rightarrow p_1) \rightarrow (\langle \rangle p_0 \rightarrow \langle \rangle p_1)) \\
\langle \rangle(\neg p_0 \vee p_1) \wedge (\langle \rangle p_0 \wedge \langle \rangle \neg p_1) \\
\langle \rangle(\neg p_0 \vee p_1); (\langle \rangle p_0 \wedge \langle \rangle \neg p_1) \\
\langle \rangle(\neg p_0 \vee p_1); \langle \rangle p_0; \langle \rangle \neg p_1 \\
\neg p_0 \vee p_1; p_0; \neg p_1 \\
\neg p_0; p_0; \neg p_1 \quad | \quad p_1; p_0; \neg p_1
\end{align*}
\]

There is a closed $\mathbf{K}$-tableau for $\neg(\langle \rangle(p_0 \rightarrow p_1) \rightarrow (\langle \rangle p_0 \rightarrow \langle \rangle p_1))$
Examples of Tableau

\[
\begin{align*}
\text{(id)} & \quad \frac{p; \neg p; X}{X} \\
\wedge & \quad \frac{\varphi \land \psi; X}{\varphi; \psi; X} \\
\vee & \quad \frac{\varphi \lor \psi; X}{\varphi; X \mid \psi; X} \\
\langle \rangle \text{K} & \quad \frac{\langle \rangle \varphi; [\!\!\! X; Z \neg \psi. \square \psi \notin Z}
\end{align*}
\]

\[
\begin{align*}
\neg([\!\!\! p_0 \rightarrow p_0)] & \quad \text{nff} \\
\hline
([\!\!\! p_0] \land \neg p_0) & \quad ([\!\!\! p_0 \land (\langle \rangle \neg p_0)] \\
\hline
([\!\!\! p_0); \neg p_0 & \quad ([\!\!\! p_0); \langle \rangle \neg p_0)] \\
\hline
\langle \rangle \text{K} & \quad \frac{\langle \rangle \neg p_0}{\langle \rangle \text{K}} \\
\hline
\neg p_0 & \quad (\langle \rangle \text{K})
\end{align*}
\]

There is no closed \textbf{K}-tableau for \(\neg([\!\!\! p_0 \rightarrow p_0)\)

There is no closed \textbf{K}-tableau for \(\neg([\!\!\! p_0 \rightarrow [\!\!\! [\!\!\! p_0))\)

How can we be sure, we only looked at one \textbf{K}-tableau for each?
Some Proof Theory

\[(id) \quad \frac{\varphi; \neg \varphi; X}{X} \quad (\land) \quad \frac{\varphi \land \psi; X}{\varphi; \psi; X} \quad (\lor) \quad \frac{\varphi \lor \psi; X}{\varphi; X \mid \psi; X} \quad (\langle \rangle K) \quad \frac{\langle \rangle \varphi; [] X; Z}{\varphi; X} \quad \forall \psi. [] \psi \notin Z\]

**Weakening:** Lemma 15  If \( \varphi; X \) has a closed \( K \)-tableau then so does \( \varphi; X; Y \) for all multisets \( Y \) (adding junk does not destroy closure)

**Inversion \( \land \):** Lemma 16  If \( \varphi \land \psi; X \) has a closed \( K \)-tableau then so does \( \varphi; \psi; X \) (applying \( (\land) \) cannot destroy closure)

**Inversion \( \lor \):** Lemma 17  If \( \varphi \lor \psi; X \) has a closed \( K \)-tableau then so do \( \varphi; X \) and \( \psi; X \) (applying \( (\lor) \) cannot destroy closure)

**Inversion fails for \( (\langle \rangle K) \):**

\[
\frac{\langle \rangle (p \lor \neg p); (q \land \neg q)}{p \lor \neg p}
\]

\[\leftarrow \text{has closed } K\text{-tableau}\] 

\[\leftarrow \text{has no closed } K\text{-tableau}\]

**Contraction:** Lemma 18  \( \varphi; X \) has a closed \( K \)-tableau iff \( \varphi; \varphi; X \) has a closed \( K \)-tableau.

*Can treat multisets as sets and vice-versa!*
Soundness of Modal Tableaux W.R.T. $\mathcal{K}$-satisfiability

A multiset of formulae $Y$ is $\mathcal{K}$-satisfiable iff there is some Kripke model $\langle W, R, \vartheta \rangle$ and some $w \in W$ with $w \models Y$ i.e. $\forall \varphi \in Y. w \models \varphi$.

Lemma 19 (id) The multiset $p; \neg p; X$ is never $\mathcal{K}$-satisfiable.

Lemma 20 ($\land$) If $\varphi \land \psi; X$ is $\mathcal{K}$-satisfiable then $\varphi; \psi; X$ is $\mathcal{K}$-satisfiable.

Lemma 21 ($\lor$) If $\varphi \lor \psi; X$ is $\mathcal{K}$-satisfiable then $\varphi; X$ is $\mathcal{K}$-satisfiable or $\psi; X$ is $\mathcal{K}$-satisfiable.

Lemma 22 ($\langle \rangle$) If $\langle \rangle \varphi; \langle \rangle X; Z$ is $\mathcal{K}$-satisfiable then $\varphi; X$ is $\mathcal{K}$-satisfiable.

Proof: Suppose $\langle \rangle \varphi; \langle \rangle X; Z$ is $\mathcal{K}$-satisfiable.

i.e. exists Kripke model $\langle W, R, \vartheta \rangle$ and some $w \in W$ with $w \models \langle \rangle \varphi; \langle \rangle X; Z$

i.e. exists Kripke model $\langle W, R, \vartheta \rangle$ and some $v \in W$ with $wRv$ and $v \models \varphi$

i.e. $v \models \varphi$ and $v \models X$

i.e. $(\varphi; X)$ is $\mathcal{K}$-satisfiable. (transitional)
Soundness of Modal Tableaux

**Theorem 8** If there is a closed $\mathcal{K}$-tableau for $Y$ then $Y$ is not $\mathcal{K}$-satisfiable.

**Proof:** Suppose there is a closed $\mathcal{K}$-tableau for $\text{nff } Y$. Proceed by induction on length of $\mathcal{K}$-tableau, recall that $\models (\land Y) \iff (\land \text{nff } Y)$.

$l = 0$: So $\text{nff } Y$ is an instance of (id). But $p; \neg p; X$ is never $\mathcal{K}$-satisfiable.

**Ind. Hyp. :** Theorem holds for all derivations of length less than some $k > 0$.

**Ind. Step:** Then $\text{nff } Y$ has a closed $\mathcal{K}$-tableau of length $k$. Top-most rule?

$(\langle \rangle \mathcal{K})$: So the top-most rule application is an instance of the $(\langle \rangle \mathcal{K})$-rule.

$\varphi; X$ has closed $\mathcal{K}$-tableau \hspace{1cm} \text{By IH. } \varphi; X$ is not $\mathcal{K}$-satisfiable.

Lemma 22: if $\langle \varphi; [] X ; Z \rangle$ is $\mathcal{K}$-satisfiable then $\varphi; X$ is $\mathcal{K}$-satisfiable.

Hence $Y = (\langle \varphi; [] X ; Z \rangle)$ cannot be $\mathcal{K}$-satisfiable.

**Corollary 2** If $\{\neg \varphi\}$ has a closed $\mathcal{K}$-tableau then $\emptyset \models \varphi$
Downward Saturated Or Hintikka Sets

A set $Y$ is downward-saturated or an Hintikka set iff:

\[ \neg: \quad \neg\neg\varphi \in Y \quad \Rightarrow \quad \varphi \in Y \]

\[ \land: \quad \varphi \land \psi \in Y \quad \Rightarrow \quad \varphi \in Y \text{ and } \psi \in Y \]

\[ \lor: \quad \varphi \lor \psi \in Y \quad \Rightarrow \quad \varphi \in Y \text{ or } \psi \in Y \]

\[ \rightarrow: \quad \varphi \rightarrow \psi \in Y \quad \Rightarrow \quad \varphi \notin Y \text{ or } \psi \in Y \]

Downward-saturated set is consistent if it does not contain \{\varphi, \neg\varphi\}, for any \varphi.

Don’t need maximality: it is not demanded that \( \forall \varphi. \varphi \in Y \text{ or } \neg\varphi \in Y \). (Hintikka)
Model Graphs

A $K$-model-graph for set $Y$ is a pair $\langle W, \triangleleft \rangle$ where $W$ is a non-empty set of downward-saturated and consistent sets, some $w_0 \in W$ contains $Y$, and $\triangleleft$ is a binary relation over $W$ such that for all $w$:

$\langle \rangle: \langle \rangle \varphi \in w \Rightarrow (\exists v \in W. w \triangleleft v \land \varphi \in v)$

$[]: [] \varphi \in w \Rightarrow (\forall v \in W. w \triangleleft v \Rightarrow \varphi \in v)$.

**Lemma 23 (Hintikka)** *If there is a $K$-model-graph $\langle W, \triangleleft \rangle$ for set $Y$ then $Y$ is $K$-satisfiable.*

Proof: Let $\langle W, R, \vartheta \rangle$ be the model where $R = \triangleleft$ and $\vartheta(w, p) = t$ iff $p \in w$. By induction on the length of a formula $\varphi$, show that $\vartheta(w, \varphi) = t$ iff $\varphi \in w$. Since $Y \subseteq w_0$ we have $w_0 \models Y$. 


Creating Downward-Saturated and Consistent Sets

**Lemma 24** If every $K$-tableau for $Y$ is open, then $Y$ can be extended into a downward-saturated and consistent $Y^*$ so every $K$-tableau for $Y^*$ is also open.

Proof: Suppose no $K$-tableau for $Y$ closes. Now consider the following systematically constructed $K$-tableau.

Stage 0: Let $w_0 = Y$.

Stage 1: Apply static rules giving finite open branch of nodes $w_0, w_1, \ldots, w_k$.

Let $Y^*$ be the multiset-union of $w_0, \ldots, w_k$.

Claim: $Y^*$ is downward-saturated (obvious) and consistent, and $Y \subseteq Y^*$.

By Contraction Lemma 18, we know $\varphi; X$ has (no) closed $K$-tableau iff $\varphi; \varphi; X$ has (no) closed $K$-tableau. (adding copies cannot affect closure)

Tableau for $Y^*$ cannot close since construction of $Y^*$ just adds back the principal formulae of each static rule application. can treat $Y^*$ as a set!
Completeness and Decidability

Lemma 25  If no $K$-tableau for $Y$ is closed, there is a $K$-model-graph for $Y$.

Proof: Suppose no $K$-tableau for $Y$ closes. Now consider the following systematic procedure

Stage 0: Let $w = Y$.

Stage 1: Apply static rules giving downward-saturated and consistent node $w^*$ (Lemma 24)

Stage 2: Let $\langle \varphi_1, \varphi_1, \cdots, \varphi_n \rangle$ be all the $\langle \rangle$-formulae in the current node.
So the current node looks like: $\langle \varphi_i; []X; Z_i \rangle$ for each $i = 1 \cdots n$.

For each $i = 1 \cdots n$ apply: $(\langle \rangle) \quad \frac{\langle \varphi_i; []X; Z_i \rangle}{\varphi_i; X} \quad \leftarrow w^* \leftarrow v_i$

Repeat Stages 1 and 2 on each node $v_i = (\varphi_i; X)$, and so on ad infinitum.

Each $(\langle \rangle)$-rule application reduces maximal-modal degree, giving termination.
Let $W$ be set of all $^*$-nodes, let $w^* \prec v_i^*$ \quad \langle W, \prec \rangle$ is a $K$-model-graph for $Y$. 

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Decidability and Analytic Superformula Property

Subformula property: the nodes (sets) of a $K$-tableau for $Y$ (i.e. $\text{nnf } Y$) only contain formulae from $\text{nnf } Y$.

Subformula property will hold if all rules simply break down formulae or copy formulae across.

Analytic superformula property: the nodes (sets) of a $L$-tableau for $Y$ (i.e. $\text{nnf } Y$) only contain formulae from a finite set $Y'$ computable from $\text{nnf } Y$ (but possibly larger than $\text{nnf } Y$).

Analytic superformula property will hold if all rules that build up formulae cannot be applied ad infinitum.

The main skill in tableau calculi is to invent rules with the subformula property or the analytic superformula property!
Completeness W.R.T. $\mathcal{K}$-Satisfiability

**Theorem 9** If there is no closed $\mathcal{K}$-tableau for $Y$ then $Y$ is $\mathcal{K}$-satisfiable.

Proof: Suppose every $\mathcal{K}$-tableau for $Y$ is open.

Use Lemma 25 to construct a $\mathcal{K}$-model-graph $\langle W, \triangleleft \rangle$ for $Y$.

For all $w \in W$, let $\vartheta(w, p) = t$ iff $p \in w$.

Then $\langle W, \triangleleft, \vartheta \rangle$ contains a world $w_0$ with $w_0 \models Y$ by Hintikka’s Lemma 23.

**Corollary 3** If there is no closed $\mathcal{K}$-tableau for $\{\neg \varphi\}$ then $\not\models \varphi$.

**Corollary 4** There is a closed $\mathcal{K}$-tableau for $Y$ iff $Y$ is not $\mathcal{K}$-satisfiable.

**Corollary 5** There is a closed $\mathcal{K}$-tableau for $\{\neg \varphi\}$ iff $\varphi$ is $\mathcal{K}$-valid.
What About Logical Consequence: a concrete example

Write $\Gamma \vdash^T \varphi : \text{iff there is a closed } \mathbf{K}\text{-tableau for } (\Gamma; \neg \varphi) \quad \text{i.e. } \text{nnf } (\Gamma; \neg \varphi)$

Want Completeness: $\Gamma \not\vdash^T \varphi \Rightarrow \exists M. M \models \Gamma \& M \not\models \varphi$

Consider: $\Gamma := \{p_0\}$ and $\varphi := []p_1$.

Then \text{nnf } (\Gamma; \neg \varphi) has only one (open) $\mathbf{K}$-tableau:

\[
\begin{array}{c}
(\Gamma; \neg \varphi) \\
\hline
(p_0; \neg []p_1) \\
(p_0; \langle \rangle \neg p_1) \\
\neg p_1
\end{array}
\]

\text{nnf}

$w_0 = \{p_0, \langle \rangle \neg p_1\} \quad w_1 = \{\neg p_1\} \quad w_0 R w_1$

Problem: although $w_0 \models \Gamma$, we don’t have $w_1 \models \Gamma$. So $M \not\models \varphi$ but $M \not\models \Gamma$.

If only we could make $w_1$ force $\Gamma$ too ...
Regaining Completeness WRT Logical Consequence

Change \((\langle\rangle)\) rule from \((\langle\rangle)\) \[
\frac{\langle\varphi; \Box X; Z\rangle \forall \psi. \Box \psi \not\in Z}{\varphi; X}
\]
to:

Transitional Rule: \((\langle\rangle \Gamma)\) \[
\frac{\langle\varphi; \Box X; Z\rangle \forall \psi. \Box \psi \not\in Z}{\varphi; X; \text{nnf } \Gamma}
\]

\((R\text{-successor forces } \Gamma)\)

Semantic reading:

if numerator is \(L\)-satisfiable in a model that forces \(\Gamma\)
then some denominator is \(L\)-satisfiable in a model that forces \(\Gamma\)\n
Stage 2: For each \(i = 1 \cdots n\) apply: \((\langle\rangle \Gamma)\) \[
\frac{\langle\varphi_i; \Box X; Z_i\rangle \varphi_i; X; \text{nnf } \Gamma}{\leftarrow w^* \leftarrow v_i \supset \text{nnf } \Gamma}
\]

By completeness: \(\Gamma \not\vdash^T \varphi\) : \iff (\exists M. \exists w. M \vdash \Gamma \& w \vdash (\Gamma; \neg \varphi))

\iff (\exists M. M \vdash \Gamma \& M \not\vDash \varphi) \iff \Gamma \not\vDash \varphi

But there is a slight problem ... (TINSTAAFL)
Regaining Decidability

Problem: K-tableau can now loop for ever: \( \Gamma := \{\langle p_0 \rangle\} \), and \( \varphi := p_1 \):

\[
\begin{align*}
(\Gamma; \neg \varphi) \quad & \quad \text{ (nnf )} \\
(\langle p_0 \rangle; \neg p_1) \quad & \quad \text{ (\langle \Gamma \rangle)} \\
(p_0; \langle p_0 \rangle) \quad & \quad \text{ (\langle \Gamma \rangle)} \\
(p_0; \langle p_0 \rangle) \quad & \quad \text{ (\langle \Gamma \rangle)} \\
\ldots
\end{align*}
\]

Solution: if we ever see a repeated node, just add a \( \Diamond \)-edge back to previous copy on path from current node to root.
Other Normal Modal Logics

$\textbf{KT}$: Static Rules: (id), $(\land)$, $(\lor)$, plus $(T)$

$$\frac{[]\varphi; X}{\varphi; ([]\varphi)^*; X} \quad \text{[]}\varphi \text{ unstarred}$$

Transitional Rule: $([]\Gamma)

$$\frac{\varphi; []X^*; Z}{\varphi; X; \text{nnf } \Gamma} \quad \forall \psi. []\psi \notin Z \quad \text{(unstar all } []\text{-formulae)}$$

$\textbf{K4}$: Static Rules: (id), $(\land)$, $(\lor)$

Transitional Rule: $([]\Gamma 4)$

$$\frac{\varphi; []X; Z}{\varphi; X; []X; \text{nnf } \Gamma} \quad \forall \psi. []\psi \notin Z$$

$\textbf{KT4}$: Static Rules: (id), $(\land)$, $(\lor)$, $(T)$

Transitional Rule: $([]\Gamma T4)$

$$\frac{\varphi; []X^*; Z}{\varphi; []X; \text{nnf } \Gamma} \quad \forall \psi. []\psi \notin Z \quad \text{(unstar all } []\text{-formulae)}$$
Examples of KT-Tableau

KT: Static Rules: (id), (\(\land\)), (\(\lor\)), plus (\(T\))

\[
\frac{[]\varphi; X}{\varphi; ([]\varphi)^*; X} \quad [\varphi \text{ unstarred}}
\]

Transitional Rule: (\(\langle\rangle\) \(\Gamma\))

\[
\frac{\langle\rangle \varphi; []X^*; Z}{\phi; X; \text{nnf} \Gamma \forall \psi. [\psi \notin Z} \quad (\text{unstar all } [\cdot]-\text{formulae})
\]

\[
\begin{align*}
\neg([p_0 \rightarrow p_0) \\
\quad \text{nnf} \\
([p_0) \land \neg p_0 \\
\quad \neg p_0 \\
\quad ([p_0); \neg p_0 \\
\quad (T) \\
\quad p_0, ([p_0)^*; \neg p_0 \\
\quad \times
\end{align*}
\]

There is a closed KT-tableau for \(\neg([p_0 \rightarrow p_0) \quad \text{i.e. } \emptyset \vdash_{KT} [p_0 \rightarrow p_0

Starring stops infinite sequence of T-rule applications.
Examples of $K4$-Tableau

$K4$: Static Rules: (id), $(\land)$, $(\lor)$

Transitional Rule: $((\langle\rangle \Gamma 4) \quad \frac{\langle\rangle \varphi; []X; Z \; \varphi; X; []X; \text{nnf } \Gamma}{\forall \psi. [] \psi \not\in Z}$

$\neg([]p_0 \rightarrow [][]p_0)$

$\frac{([]p_0) \land (\langle\rangle \langle\rangle \neg p_0)}{(\langle\rangle \Gamma 4)}$

$\frac{[]p_0; \langle\rangle \langle\rangle \neg p_0}{(\langle\rangle \Gamma 4)}$

$\frac{p_0; []p_0; \langle\rangle \neg p_0}{p_0; [ ] p_0; [ ] p_0}$

$\frac{p_0; []p_0; \neg p_0}{p_0; [ ] p_0}$

There is closed $K4$-tableau for $\neg([]p_0 \rightarrow [][]p_0)$ i.e. $\emptyset \vdash_{K4} []p_0 \rightarrow [][]p_0$

Need loop check: $K4$-tableau for $(\langle\rangle p_0; []\langle\rangle p_0)$ has infinite branch.
Follow The Procedure ...

Prove Weakening.

Prove Inversion for all Static Rules.

Check if Transitional Rule has Inversion (unlikely).

Prove Soundness: If there is a closed $\textbf{KL}$-tableau for $Y$ then $Y$ is not $\textbf{KL}$-satisfiable.

Define appropriate notion of $\textbf{L}$-model-graph.

Prove Hintikka’s Lemma: If there is an $\textbf{L}$-model-graph for $Y$ then $Y$ is $\textbf{KL}$-satisfiable.

Prove Completeness: If there is no closed $\textbf{KL}$-tableau for $Y$ then $Y$ is $\textbf{KL}$-satisfiable.

Add changes to transitional rule(s) for handling $\Gamma \vdash^T_L \varphi$

Prove termination (by analytic superformula property and tracking of loops).
Soundness for Rule \( \langle \rangle T4 \)

Example: \( \langle \rangle T4 \)  
\[
\frac{\langle \varphi; [] X^*; Z \rangle}{\varphi; [] X} \quad \forall \psi. [] \psi \not\in Z
\]

All depends upon:

Lemma: if \( \langle \rangle \varphi; []; X; Z \) is \( \mathcal{KT}4 \)-satisfiable then \( \varphi; X \) is \( \mathcal{KT}4 \)-satisfiable.

Proof: Suppose \( \langle \rangle \varphi; []; X; Z \) is \( \mathcal{KT}4 \)-satisfiable.

i.e. exists transitive Kripke model \( \langle W, R, \vartheta \rangle \) and some \( w \in W \) with \( w \models \langle \rangle \varphi; []; X; Z \)

i.e. exists transitive Kripke model \( \langle W, R, \vartheta \rangle \) and some \( v \in W \) with \( wRv \) and \( v \models (\varphi; X; [] X) \quad ([] X \rightarrow [] [] X) \)

i.e. exists transitive Kripke model \( \langle W, R, \vartheta \rangle \) and some \( v \in W \) with \( wRv \) and \( v \models (\varphi; [] X) \) can regain \( X \) by \( T \) rule
Tableaux Versus Hilbert Calculi

**Algorithm:** Systematic procedure gives algorithm for finding (closed) tableaux.

**Decidability:** easier than in Hilbert Calculi.

**Modularity:** Must invent new rules for new axioms. Reuse completeness proof based upon systematic procedure with tweaks. Rules require careful design to regain decidability e.g. starring, looping, dynamic looping etc.

**Automated Deduction:** Logics WorkBench [http://www.lwb.unibe.ch](http://www.lwb.unibe.ch) has implementation of tableau theorem provers for many fixed logics e.g. K, KT, K4, KT4, ...

**Automated Deduction:** The Tableaux WorkBench [http://arp.anu.edu.au/~abate/twb](http://arp.anu.edu.au/~abate/twb) provides a way to implement tableau theorem provers for any tableau calculus that fits its syntax e.g. KD45, KtS4, Int, IntS4, ...
Tense Logics: interpret $[\varphi]$ as “$\varphi$ is true always in the future”.

$W$ represents moments of time

$R$ captures the flow of time

Temporal Logics: similar, but use a more expressive binary modality $\varphi U \psi$ to capture “$\varphi$ is true at all time points from now until $\psi$ becomes true”.

Shall look at Syntax, Semantics, Hilbert and Tableau Calculi.
Tense Logics: Syntax and Semantics

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \langle F \rangle \varphi \mid [F] \varphi \mid \langle P \rangle \varphi \mid [P] \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \)

Boolean connectives interpreted as for modal logic.

Given some Kripke model \( \langle W, R, \vartheta \rangle \) and some \( w \in W \), we compute the truth value of a non-atomic formula by recursion on its shape:

\[
\vartheta(w, \langle F \rangle \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at some } v \in W \text{ with } wRv \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, [F] \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at every } v \in W \text{ with } wRv \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, \langle P \rangle \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at some } v \in W \text{ with } vRw \\
  f & \text{otherwise}
\end{cases}
\]

\[
\vartheta(w, [P] \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at every } v \in W \text{ with } vRw \\
  f & \text{otherwise}
\end{cases}
\]
Tense Logics: Syntax and Semantics

\[ \vartheta(w, \langle F \rangle \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at some } v \in W \text{ with } w R v \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(w, [F] \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at every } v \in W \text{ with } w R v \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(w, \langle P \rangle \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at some } v \in W \text{ with } v R w \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(w, [P] \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at every } v \in W \text{ with } v R w \\
  f & \text{otherwise}
\end{cases} \]

Example: If \( W = \{w_0, w_1, w_2\} \) and \( R = \{(w_0, w_1), (w_0, w_2)\} \) and \( \vartheta(w_1, p_3) = t \) then \( \langle W, R, \vartheta \rangle \) is a Kripke model as pictured below:

\[ \vartheta(w_0, \langle F \rangle p_3) = t \]
\[ \vartheta(w_2, \langle P \rangle \langle F \rangle p_3) = t \]
\[ \vartheta(w_0, [P] p_1) = t \]

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Hilbert Calculus for Modal Logic $K_t$

**Axiom Schemata:** Axioms for PC plus:

$K[F]: [F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)$

$K[P]: [P](\varphi \to \psi) \to ([P]\varphi \to [P]\psi)$

$FP: \varphi \to [F]\langle P \rangle \varphi$

$PF: \varphi \to [P]\langle F \rangle \varphi$

**Rules of Inference:**

(Ax) $\Gamma \vdash \varphi$ is an instance of an axiom schema

(Id) $\Gamma \vdash K_t \varphi$ $\varphi \in \Gamma$

(MP) $\Gamma \vdash K_t \varphi$ $\Gamma \vdash K_t \varphi \to \psi$ $\Gamma \vdash K_t \psi$

(Nec$[F]$) $\Gamma \vdash K_t \varphi$ $\Gamma \vdash K_t [F]\varphi$

(Nec$[P]$) $\Gamma \vdash K_t \varphi$ $\Gamma \vdash K_t [P]\varphi$

Soundness, Completeness, Correspondence etc. : Let $K_t \equiv K$ be class of all Kripke Tense frames

$\Gamma \vdash K_t A_1, A_2, \ldots, A_n \varphi$ iff $\Gamma \models K_t A_1, A_2, \ldots, A_n \varphi$
Different Models of Time

Arbitrary Time: $K_t$

Reflexive Time: $\varphi \rightarrow \langle F \rangle \varphi$  
Transitive Time: $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$

Dense Time: $\langle F \rangle \varphi \rightarrow \langle F \rangle \langle F \rangle \varphi$  
Never Ending Time: $[F] \varphi \rightarrow \langle F \rangle \varphi$

Backward Linear: $\langle F \rangle \langle P \rangle \varphi \rightarrow \langle P \rangle \varphi \lor \varphi \lor \langle F \rangle \varphi$

Forward Linear: $\langle P \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi \lor \varphi \lor \langle P \rangle \varphi$


Discrete $\langle \mathbb{Z}, < \rangle$, Rational $\langle \mathbb{Q}, < \rangle$, Real $\langle \mathbb{R}, < \rangle$ linear and non-reflexive models of time also possible: see Goldblatt.

Tableau-like calculi exist: see Mosaic Method
PLTL: Propositional Linear Temporal Logic

Atomic Formulae: \( p \ ::=  p_0 \mid p_1 \mid p_2 \mid \cdots \)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \boxplus \varphi \mid [F] \varphi \mid \langle F \rangle \varphi \mid \varphi \cup \psi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \)

Boolean connectives interpreted as for modal logic.

Linear Time Kripke Model: \( \langle S, \sigma, R, \vartheta \rangle \)

\( S \): non-empty set of states

\( \sigma \): \( \mathbb{N} \rightarrow S \) enumerates \( S \) as sequence \( \sigma_0, \sigma_1, \cdots \) with repetitions when \( S \) finite

\( \vartheta \): \( S \times \text{Atm} \rightarrow \{ t, f \} \)

\( R \): is a binary relation over \( S \)

**Condition:** \( R = \sigma^* \) \hspace{1cm} (\( R \) is the reflexive and transitive closure of \( \sigma \))
Semantics of PLTL

\[ \vartheta(s_i, \bigoplus \varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_{i+1}, \varphi) = t \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, (F)\varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_j, \varphi) = t \text{ for some } j \geq i \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, [F]\varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_j, \varphi) = t \text{ for all } j \geq i \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, \varphi U \psi) = \begin{cases} 
  t & \text{if } \exists k \geq i. \vartheta(s_k, \psi) = t \land \forall j. i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = t \\
  f & \text{otherwise}
\end{cases} \]

\[ s_i \quad s_{i+1} \quad \cdots \quad s_j \quad \cdots \quad s_k \]

\[ p U q \quad p, \neg q \quad \cdots \quad p, \neg q \quad \cdots \quad q \]

Note: when \( k \neq i \), the state \( s_k \) is the first state after \( s_i \) where \( q \) is true.
Semantics of PLTL

\[ \vartheta(s_i, \bigoplus \varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_{i+1}, \varphi) = t \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, (F)\varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_j, \varphi) = t \text{ for some } j \geq i \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, [F]\varphi) = \begin{cases} 
  t & \text{if } \vartheta(s_j, \varphi) = t \text{ for all } j \geq i \\
  f & \text{otherwise}
\end{cases} \]

\[ \vartheta(s_i, \varphi U \psi) = \begin{cases} 
  t & \text{if } \exists k \geq i. \vartheta(s_k, \psi) = t \land \forall j. i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = t \\
  f & \text{otherwise}
\end{cases} \]

\[ s_i \quad s_{i+1} \quad \cdots \quad s_j \quad \cdots \quad s_k \]

\[ \neg(p U q), \neg q \quad \neg q \quad \cdots \quad \neg q \quad \cdots \quad \neg q \]

\[ \neg(p U q) \quad \neg q \quad \cdots \quad \neg p, \neg q \quad \cdots \quad q \]

\( q \) is always false, or

\( p \) false before \( q \) true

Note: when \( k \neq i \), the state \( s_k \) is the first state after \( s_i \) where \( q \) is true. And \( p \) is false in some \( s_j \) before state \( s_k \).
Hilbert Calculus for PLTL

Axiom Schemata: axioms for PC plus

\( \mathbf{K}[F] \): \([F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)\)

\( \mathbf{K} \oplus \): \(\oplus(\varphi \to \psi) \to (\oplus \varphi \to \oplus \psi)\)

\( \text{Fun:} \) \(\oplus \neg \varphi \leftrightarrow \neg \oplus \varphi\)

\( \text{Mix:} \) \([F]\varphi \to (\varphi \land \oplus [F]\varphi)\)

\( \text{Ind:} \) \([F](\varphi \to \oplus \varphi) \to (\varphi \to [F]\varphi)\)

\( \mathcal{U}_1 \): \(\varphi \mathcal{U} \psi \to \langle F \rangle \psi\)

\( \mathcal{U}_2 \): \(\varphi \mathcal{U} \psi \leftrightarrow \psi \lor (\neg \psi \land \varphi \land \oplus (\varphi \mathcal{U} \psi))\)

\(\text{Rules:}\) (Id), (Ax), MP and (Nec\([F]\)) and (Nec\(\oplus\))
Tableau Calculus for PLTL

Presence of Induction Axiom Ind means no finitary cut-free sequent calculus (must guess induction hypothesis)

Requires two pass method: build a model-graph, check that it contains a model.

Use the rewriting steps from proof of Lemma 14 for non-modal connectives and rewriting steps below rather than explicit NNF rules:

\[ \neg \oplus \varphi \rightarrow \oplus \neg \varphi \quad \neg [F] \varphi \rightarrow \langle F \rangle \neg \varphi \quad \neg \langle F \rangle \varphi \rightarrow [F] \neg \varphi \]

\[ \langle F \rangle \varphi \rightarrow (\varphi \lor \oplus \langle F \rangle \varphi) \quad [F] \varphi \rightarrow (\varphi \land \oplus [F] \varphi) \]

\[ (\varphi U \psi) \rightarrow \psi \lor (\varphi \land \oplus (\varphi U \psi)) \quad \neg (\varphi U \psi) \rightarrow \neg \psi \land (\neg \varphi \lor \oplus \neg (\varphi U \psi)) \]

Static Rules: (id) \[ \frac{p; \neg p; X}{X} \]

(\land) \[ \frac{\varphi \land \psi; X}{\varphi; \psi; X} \]

(\lor) \[ \frac{\varphi \lor \psi; X}{\varphi; X \mid \psi; X} \]

Transitional Rule: (\oplus) \[ \frac{\oplus \varphi; \oplus X; Z}{\varphi; X} \quad Z \text{ contains only atoms & negated atoms} \]
Tableau Calculus for PLTL: Pass 1

\[
\begin{align*}
\frac{p; \neg p; X}{\times} & \quad \frac{\phi \land \psi; X}{\phi; \psi; X} \quad \frac{\phi \lor \psi; X}{\phi; X \mid \psi; X} \quad \frac{\oplus \phi; \oplus X; Z}{\phi; X} \\
\end{align*}
\]

\(Z\) contains only atoms & negated atoms

Create a node containing \(Y\) and mark it as unexpanded

While some node \(x\) is marked unexpanded do

- if \(x\) is an instance of (id) then mark \(x\) as closed
- Else if we can apply \((\land)\) or \((\lor)\) or a rewriting step to \(x\) then do so, mark children as unexpanded, and label edges to children with rule name
- Else if \(\oplus\) rule is applicable then if some existing node \(v\) contains \(\phi; X\) then add an \(\oplus\)-edge from \(x\) to \(v\) else apply the \((\oplus)\) rule to create a \(\oplus\)-child \(y\) containing \(\phi; X\) and mark \(y\) as unexpanded
- Else make \(x\) reflexive by adding \(\oplus\)-edge from \(x\) to \(x\)
- Mark \(x\) as not unexpanded
Tableau Method for PLTL: Pass 2

A path from node \( w \) is a maximal (cyclic) sequence of nodes starting at \( w \).

A node \( w \) contains an eventuality if \( \langle F \rangle \phi \in w \) or \( \psi U \phi \in w \) for some formula \( \phi \).

A path fulfills an eventuality \( \langle F \rangle \phi \) or \( \psi U \phi \) if some node on the path contains \( \phi \).

An eventuality \( \langle F \rangle \phi \in w \) or \( \psi U \phi \in w \) is fulfilled if some path from \( w \) fulfills it.

Repeat: until no node is deleted or the root node is deleted

- Delete all closed nodes (that contain a pair \( \{p, \neg p\} \))
- Delete all nodes with no children
- Delete any node which contains an unfulfilled eventuality

If some single path from the root node fulfills all eventualities contained in its nodes then \( Y \) is PLTL-satisfiable, otherwise it is not.

That is: some single path from the root must fulfil all its eventualities itself!
Pass 1: $[F]((\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q))$

Delete all closed nodes
Pass 2: \([F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)\)

Delete nodes containing unfulfilled eventualities note maximal!
Pass 2: \([F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)\)

Delete nodes containing unfulfilled eventualities
Pass 2: \([F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)\)

Delete nodes containing unfulfilled eventualities
Pass 2: $[F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$

$[F](\neg p \land \neg q); \langle F \rangle p \lor \langle F \rangle q$ \hspace{1cm} $[F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$

$\neg p; \neg q; \oplus [F](\neg p \land \neg q); \oplus \langle F \rangle q$

$\neg p; \neg q; \oplus [F](\neg p \land \neg q); \oplus \langle F \rangle p$

Delete nodes with no children
Pass 2: $[F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$

$[F](\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$

$\neg p; \neg q; \oplus [F](\neg p \land \neg q)$

$\vdash p \lor \oplus \langle F \rangle p$

Delete nodes with no children
Pass 2: $\mathcal{F}(\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$

Root deleted so stop and declare $\mathcal{F}(\neg p \land \neg q) \land (\langle F \rangle p \lor \langle F \rangle q)$ as unsatisfiable
Extensions to Other Temporal Logics

Extends to temporal logics with a \( \ominus \) operator and a \( \varphi S \psi \) operator.

Extends to branching-temporal logics.

Usually requires space exponential in size of \( Y \) since we cannot follow each branch individually but must construct the whole state space for \( Y \) first.

Two pass nature compiles away the induction hypothesis.
Further Reading


Logics of Time and Computation. R. I. Goldblatt CSLI Lecture Notes Number 7, Center for the Study of Language and Information, Stanford, 1987

Modal Logic. P Blackburn, M de Rijke and Y Venema Cambridge University Press


