Classical First-Order Logic - Answers to Exercises 2
Overview of Logic and Computation
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Exercise 1. Consider the following formulae:

- $\forall x.(\neg p(x, x))$
- $\forall x.\exists y.(p(x, y))$
- $\forall x.\forall y.\forall z.((p(x, y) \land p(y, z)) \rightarrow p(x, z))$
- $\forall x.\forall y.(p(x, y) \rightarrow \exists z.(p(x, z) \land p(z, y)))$

Find a satisfying interpretation for these formulae. Argue why every interpretation satisfying these formulae has an infinite domain.

Answer. The interpretation $I = (D = \mathbb{R}, \alpha)$, where: $\alpha_2[p] : (n, m) \mapsto \begin{cases} true & \text{if } n < m \\ false & \text{otherwise} \end{cases}$ satisfies all of the formulae above. We give an intuitive idea of the reason why it is the case. We have that $I \models \forall x.(\neg p(x, x))$ essentially as for every $n \in \mathbb{R}$, $n \neq n$. $I \models \forall x.\exists y.(p(x, y))$ as for every $n \in \mathbb{R}$ there is a $m \in \mathbb{R}$ such that $n < m$ (you can take $n + 1$). $I \models \forall x.\forall y.\forall z.((p(x, y) \land p(y, z)) \rightarrow p(x, z))$ as for every $n, m, o \in \mathbb{R}$ we get that if $n < m$ and $m < o$ then $n < o$. $I \models \forall x.\forall y.(p(x, y) \rightarrow \exists z.(p(x, z) \land p(z, y)))$ as for every $n, m \in \mathbb{R}$ if $n < m$ then there is an element $o \in \mathbb{R}$ such that $n < o$ and $o < m$.

The satisfying interpretations of this set of formulae must be infinite. Let $I = (D, \alpha)$ be such an interpretation. First we know that $D$ is by definition not empty. So there is at least one element $d \in D$. But then there must be an element $d_0 \in D$ such that $\alpha[p](d, d_0) = true$, as $I \models \forall x.\exists y.(p(x, y))$. We know that this element $d_0$ cannot be identical to $d$ as $I \models \forall x.(\neg p(x, x))$ and thus $\alpha[p](d, d) = false$. So we have two different elements $d, d_0$ such that $\alpha[p](d, d_0) = true$. Thus there is an element $d_1 \in D$ such that $\alpha[p](d, d_1) = true$ and $\alpha[p](d_1, d_0) = true$ as $I \models \forall x.\forall y.(p(x, y) \rightarrow \exists z.(p(x, z) \land p(z, y)))$. Once again we can show that $d_0 \neq d_1 \neq d$ as $I \models \forall x.(\neg p(x, x))$.

We can easily apply again the previous steps to the new elements created, and this process never stops. Thus we get that $D$ is infinite.

Exercise 2. We say that a term $t$ is linear if for every variable $x \in \text{var}(t)$ we have that $x$ occurs only once in $t$. Show by induction on the structure of terms that if $t$ is a linear term and $\sigma$ is a substitution such that $\text{var} \text{cod}(\sigma) = \emptyset$, then $t \sigma$ is linear.

Answer. Let $t$ be a term and $\sigma$ a substitution. Assume that $t$ is linear and assume that $\text{var} \text{cod}(\sigma) = \emptyset$. We prove by induction on the structure of $t$ that $t \sigma$ is linear:

- Base cases: We have two base cases here:
  - Assume that $t := x$ for some variable $x$. We need to show that $t \sigma = x \sigma$ is linear. We have two cases to consider: either $x \in \text{dom}(\sigma)$ or $x \notin \text{dom}(\sigma)$. If $x \in \text{dom}(\sigma)$ then we have that $x \sigma = t_0$ for some term $t_0$ that $\sigma$ substitutes $x$ with. But as $\text{var} \text{cod}(\sigma) = \emptyset$ we get that there is no variable occurring in $t_0$, so it is linear by definition. If $x \notin \text{dom}(\sigma)$ then we have that $x \sigma = x$ by definition of substitutions, and $x$ is linear by definition. In both cases $x \sigma = t \sigma$ is linear.
- Assume that $t := c$ for some constant $c$. We need to show that $t\sigma = c\sigma$ is linear. We know that $c\sigma = c$ by definition of substitutions. By definition $c$ is linear as no variable occurs in it. So $c\sigma = t\sigma$ is linear.

- Inductive case: We have only one inductive case here. Assume that $t := f(t_1,t_2,...,t_n)$ for some terms $t_1,...,t_n$ and an $n$-ary function symbol $f$. Our induction hypothesis is: if $t_i$ is linear then $t_i\sigma$ is linear where $i \in \{1,...,n\}$. As by assumption we know that $t$ is linear we have that $f(t_1,...,t_n)$ is linear, and thus it can easily be proved that $t_i$ is linear for $i \in \{1,...,n\}$. Thus the induction hypothesis can be reduced to: $t_i\sigma$ is linear where $i \in \{1,...,n\}$. Now we show that $t\sigma = (f(t_1,...,t_n))\sigma$ is linear. First note that $(f(t_1,...,t_n))\sigma = f(t_1\sigma,...,t_n\sigma)$ by definition of substitutions. Second, let $x$ be a variable occurring in $f(t_1\sigma,...,t_n\sigma)$.

We want to show that $x$ occurs only once in $f(t_1\sigma,...,t_n\sigma)$: by proving this we know that $f(t_1\sigma,...,t_n\sigma)$ is linear and so we reach our goal. We prove the previous statement by reductio, so let us assume that $x$ does not occur only once in $f(t_1\sigma,...,t_n\sigma)$, i.e. $x$ occurs at least twice in $f(t_1\sigma,...,t_n\sigma)$. Then we have two cases:

- $x$ occurs in one single term $t_j\sigma$ at least twice for some $j \in \{1,...,n\}$. Then we have that $t_j\sigma$ is not linear, as a variable occurring in it occurs in it more than once. But this is in contradiction with our induction hypothesis that says that $t_j\sigma$ is linear.

- $x$ occurs in two different terms $t_k\sigma$ and $t_l\sigma$ for some $k,l \in \{1,...,n\}$ such that $k \neq l$. Then we deduce that $x$ is a variable appearing in both $t_k\sigma$ and $t_l\sigma$ as $\var{\text{var}} = \emptyset$: it cannot be the case that one of the occurrences of $x$ in $t_k\sigma$ or $t_l\sigma$ was “created” by the substitution as there is no variable in $\var{\text{var}}$. But then it means that $x$ occurs at least twice in $f(t_1\sigma,...,t_n\sigma)$, which implies that this term is not linear. But this is in contradiction with our hypothesis that $t := f(t_1,...,t_n)$ is linear.

In both cases we reached a contradiction, so the assumption according to which $x$ does not occur only once in $f(t_1\sigma,...,t_n\sigma)$ is contradictory. Thus we deduce that $x$ occurs only once in $f(t_1\sigma,...,t_n\sigma)$. Consequently $t\sigma = f(t_1\sigma,...,t_n\sigma)$ is linear.

**Exercise 3.** Determine if the following multisets of equations are unifiable:

1. $E_1 = \{ p(x,y) = p(z,f(x)), y = f(f(x)), f(z) = f(y) \}$
2. $E_2 = \{ f(x,y) = f(x,t), p(x,t) = p(z,y) \}$
3. $E_3 = \{ g(x,f(y),t) = g(x,y,t), p(t) = p(y), x = g(t,t,t) \}$
4. $E_4 = \{ f(h(y),x) = f(z,t), g(z) = g(h(y)), f(h(z),t) = f(h(h(y)),y) \}$

**Answer.** In the following the equalities appearing in blue are the one that are produced by the rule applied on the right of the set of equations.

1. $E_1$ is not unifiable:

   $E_1 \iff \begin{align*}
   &\{ p(x,y) = p(z,f(x)), y = f(f(x)), f(z) = f(y) \} & \text{Given} \\
   \quad \iff \{ x = z, y = f(x), y = f(f(x)), f(z) = f(y) \} & \text{Decompose} \\
   \quad \iff \{ x = z, y = f(x), f(x) = f(f(x)), f(z) = f(f(x)) \} & \text{Apply} \\
   \quad \iff \{ x = z, y = f(x), x = f(x), f(z) = f(f(x)) \} & \text{Decompose} \\
   \quad \iff \bot & \text{Occur Check}
   \end{align*}$
Exercise 4. Using the resolution calculus show that the following set clauses are
unsatisfiable:

1. \( S_1 = \{ \neg p_0(x) \lor p_1(f(z)), p_2(y) \lor \neg p_1(y), \neg p_2(f(x)) \lor p_0(x), p_0(f(t)) \} \)
2. \( S_2 = \{ r(x, y), p_0(t) \lor \neg p_1(x), p_1(f(y)) \lor \neg r(y, f(z)), \neg p_0(x) \} \)
3. \( S_3 = \{ s(x, t, f(z)) \lor p(t), r(z, f(y)) \lor \neg p(x), \neg r(t, f(x)), \neg s(x, y, x) \} \)

Answer.

1. \( S_1 \) is unsatisfiable:

\[
\begin{align*}
p_2(y) \lor \neg p_1(y) & \quad \neg p_2(f(x)) \lor \neg p_0(x) \\
\hline
\neg p_1(f(x)) & \lor \neg p_0(x) \\
\hline
\neg p_0(x) & \lor \neg p_0(x) \\
\hline
\neg p_0(x) & \lor p_1(f(z)) \\
\hline
\neg p_0(x) & \lor z \\
\hline
\neg p_0(x) & \lor p_0(f(t)) \\
\hline
p_0(f(t)) & \lor z \\
\hline
\end{align*}
\]

2. \( S_2 \) is unsatisfiable:

\[
\begin{align*}
\neg p_0(x) & \lor p_0(t) \lor \neg p_1(f(x)) \\
\hline
\neg p_1(f(t)) & \lor p_1(f(y)) \\
\hline
\neg r(z, f(t)) & \lor \neg r(z, f(y)) \\
\hline
\neg r(z, f(t)) & \lor r(x, y) \\
\hline
\end{align*}
\]
3. $S_3$ is unsatisfiable:

$$
\frac{-s(x, y, x) \quad s(x, t, f(z)) \lor p(t)}{-p(x) \quad res} \quad \frac{-r(t, f(x)) \quad r(z, f(y)) \lor \neg p(x)}{p(t) \quad res}
$$