Exercise 1. Consider the following model $M = (W, R, V)$:

![Diagram of the model]

where all the propositional variables made true at a world are explicitly written in the circle related to the world. For example, we have that $M, w_1 \models p$ and $M, w_3 \models q$ while $\not W, w_4 \not \models q$ and $M, w_1 \not \models p$. Are the following statements true or false? In each case justify your answer.

1. $M, w_0 \models \Box \Diamond q$
2. $M, w_3 \models \Box \neg p$
3. $M, w_2 \models \Box \Diamond (p \land q)$
4. $M, w_0 \models \Box (p \lor q)$
5. $M, w_3 \models \Diamond (p \rightarrow q)$
6. $M \models \Box p \rightarrow p$
7. $M \models \Diamond (\Box p \rightarrow \Diamond (p \lor \neg \Box q))$
8. $(W, R) \models \Box p \rightarrow \Diamond p$
9. $(W, R) \models \Box p \rightarrow p$

Answer.

1. $M, w_0 \models \Box \Diamond q$ holds as we have that for every $v \in W$ if $w_0 R v$, then there is a $v' \in W$ such that $v R v_0$ and $M, v' \models q$. For $w_3$ we have that $w_3 R w_3$ and $M, w_3 \not \models q$, so $M, w_3 \models \Diamond q$. For $w_2$ we have that $w_2 R w_1$ and $M, w_1 \models q$, so $M, w_2 \models \Diamond q$. For $w_1$ we have that $w_1 R w_1$ and $M, w_1 \models q$, so $M, w_1 \models \Diamond q$. So we have that $M, w_0 \models \Box \Diamond q$ as there is no other point accessible from $w_0$.

2. $M, w_3 \not \models \Box \neg p$ as $w_3 R w_4$ and $M, w_4 \models p$, hence $M, w_4 \not \models \neg p$.

3. $M, w_2 \models \Box \Diamond (p \land q)$ holds. We have that $M, w_1 \models p \land q$ as well as $w_2 R w_1$, hence $M, w_2 \models \Diamond (p \land q)$. But as $w_1 R w_2$ we get $M, w_1 \models \Diamond \Diamond (p \land q)$. As $w_1$ is the only point accessible from $w_2$ we get $M, w_2 \models \Box \Diamond (p \land q)$.
4. $\mathcal{M}, w_0 \models \Box(p \lor q)$ holds. We have that $w_0 R w_1$ and $\mathcal{M}, w_1 \models p$ hence $\mathcal{M}, w_1 \models p \lor q$. We have that $w_0 R w_2$ and $\mathcal{M}, w_2 \models p$ hence $\mathcal{M}, w_2 \models p \lor q$. We have that $w_0 R w_3$ and $\mathcal{M}, w_3 \models q$ hence $\mathcal{M}, w_3 \models p \lor q$. As there is no other point accessible from $w_0$ we get $\mathcal{M}, w_0 \models \Box(p \lor q)$.

5. $\mathcal{M}, w_3 \models \Box(p \rightarrow q)$ as $w_3 R w_5$ and $\mathcal{M}, w_3 \models q$ hence $\mathcal{M}, w_3 \models p \rightarrow q$.

6. $\mathcal{M} \models \Box p \rightarrow p$ holds. To prove it we need to show that for every $w \in W$, if $\mathcal{M}, w \models \Box p$ then $\mathcal{M}, w \models p$. The only points that force $\Box p$ are $w_1$ and $w_2$, so it is sufficient to show that they also force $p$. But this is the case, so we can conclude that $\mathcal{M} \models \Box p \rightarrow p$.

7. $\mathcal{M} \models \Box(\Box p \rightarrow (\Box \lor \Box q))$ holds as every point that has a successor is such that if it forces $\Box p$ then it forces $\Box(\Box \lor \Box q)$. The only relevant points to consider are $w_1, w_2$ as they are the only ones to have one single successor which forces $\Box p$ (the other points force trivially $\Box(\Box p \rightarrow (\Box \lor \Box q))$ as they have a successor which does not satisfy $\Box p$). We have that $\mathcal{M}, w_1 \models p$, hence $\mathcal{M}, w_1 \models (\Box \lor \Box q)$ and thus $\mathcal{M}, w_1 \models \Box p \rightarrow (\Box \lor \Box q)$. Similarly we have $\mathcal{M}, w_2 \models p$, hence $\mathcal{M}, w_2 \models (\Box \lor \Box q)$ and thus $\mathcal{M}, w_2 \models \Box p \rightarrow (\Box \lor \Box q)$. So we can conclude from this that $\mathcal{M}, w_1 \models (\Box \lor \Box q)$ and $\mathcal{M}, w_2 \models (\Box \lor \Box q)$. Thus we get $\mathcal{M} \models (\Box \lor \Box q)$.

8. $(W, R) \models \Box p \rightarrow \Box p$ holds. Take any valuation $V$ on $(W, R)$. It can be checked that for every point $w \in W$, if $(W, R, V), w \models \Box p$ then $(W, R, V), w \models \Box p$. Let us show this for $w_0$. Assume that $(W, R, V), w_0 \models \Box p$. Then we know by definition that $(W, R, V), w_1 \models p$ as $w_0 R w_1$, hence $(W, R, V), w_0 \models \Box p$. A similar argument can be provided for all the other points in $W$.

9. $(W, R) \not\models \Box p \rightarrow p$. Consider the following valuation $V$ such that for every $w \in W$ such that $w \neq w_0$, $V(w, p) = t$ and $V(w_0, p) = t$. We thus get that $(W, R, V), w_4 \models \Box p$ as $(W, R, V), w_0 \models p$, while $(W, R, V), w_1 \not\models p$. So $(W, R, V) \not\models \Box p \rightarrow p$ hence $(W, R) \not\models \Box p \rightarrow p$.

Exercise 2. For each of the following formulae determine if it is valid. If it is justify your claim. If it is not provide a countermodel and then determine if it is satisfiable. If it is provide a model. If it is not justify your claim.

1. $\Diamond(p \land q) \rightarrow (\Diamond p \land \Diamond q)$
2. $\Box(p \lor q) \rightarrow (\Box p \lor \Box q)$
3. $\Diamond(p \land \neg p)$
4. $\Diamond(p \land q) \leftrightarrow (\Box p \land \Box q)$
5. $(\neg\Diamond p \land \neg\Diamond \rightarrow p)$
6. $(\Box p \land \Box \neg p) \lor \Diamond \top$

Answer.

1. $\Diamond(p \land q) \rightarrow (\Diamond p \land \Diamond q)$ is valid. Let $\mathcal{M} = (W, R, V)$ be a model and $w \in W$. Assume that $\mathcal{M}, w \models \Diamond(p \land q)$. Then there is a $v \in W$ such that $w R v$ and $\mathcal{M}, v \models p \land q$, hence $\mathcal{M}, v \models p$ and $\mathcal{M}, v \models q$. Thus we have that there is a $v \in W$ such that $w R v$ and $\mathcal{M}, v \models p$, hence $\mathcal{M}, w \models \Diamond p$. Similarly there is a $v \in W$ such that $w R v$ and $\mathcal{M}, v \models q$, hence $\mathcal{M}, w \models \Diamond q$. So $\mathcal{M}, w \models \Diamond(p \land q)$, hence $\mathcal{M}, w \models \Diamond(p \land q) \rightarrow (\Diamond p \land \Diamond q)$. As $w$ and $\mathcal{M}$ are arbitrary we get that $\models \Diamond(p \land q) \rightarrow (\Diamond p \land \Diamond q)$.

2. $\Box(p \lor q) \rightarrow (\Box p \lor \Box q)$ is not valid. Consider the following model $\mathcal{M}$:
We have that $\mathcal{M}, w \models \Box(p \land q)$ but $\mathcal{M}, w \not\models \Box p \lor \Box q$. We neither have $\mathcal{M}, w \not\models \Box p$, as $w_0 R w_1$ and $\mathcal{M}, w_1 \not\models p$; nor $\mathcal{M}, w \not\models \Box q$, as $w_0 R w_2$ and $\mathcal{M}, w_2 \not\models q$. $\Box(p \land q) \rightarrow (\Box p \lor \Box q)$ is however satisfiable as shows the previous model: we have that $\mathcal{M}, w_1 \models \Box(p \lor q)$ and $\mathcal{M}, w_1 \models \Box p$ hence $\mathcal{M}, w_1 \models \Box p \lor \Box q$.

3. $\Diamond(p \land \neg p)$ is not valid. Consider the previous model $\mathcal{M}$. It is such that there is no $v \in W$ such that $w_1 R v$, so by definition $\mathcal{M}, w_1 \not\models \Diamond(p \land \neg p)$. It is not satisfiable either. Assume for contradiction that it is. Then there is a model $\mathcal{N} = (W, R, V)$ and a $w \in W$ such that $\mathcal{N}, w \models \Diamond(p \land \neg p)$. Thus there is a $v \in W$ such that $\mathcal{N}, v \models p \land \neg p$, hence $\mathcal{N}, v \models p$ and $\mathcal{N}, v \models \neg p$, i.e., $\mathcal{N}, v \not\models p$ but this is a contradiction.

4. $\Box(p \land q) \leftrightarrow (\Box p \land \Box q)$ is valid. Let $\mathcal{M} = (W, R, V)$ and $w \in W$. Assume that $\mathcal{M}, w \models \Box(p \land q)$. Let $v \in W$ such that $w R v$. As $\mathcal{M}, w \models \Box(p \land q)$ we get that $\mathcal{M}, v \models p \land q$ hence $\mathcal{M}, v \models p$ and $\mathcal{M}, v \models q$. As $v$ is arbitrary we get that $\mathcal{M}, w \models \Box p \land \Box q$, hence $\mathcal{M}, w \models \Box(p \land q) \rightarrow (\Box p \land \Box q)$. Now assume that $\mathcal{M}, w \models \Box p \land \Box q$, hence $\mathcal{M}, w \models \Box p$ and $\mathcal{M}, w \models \Box q$. Let $v \in W$ such that $w R v$. As $\mathcal{M}, w \models \Box p$ we get $\mathcal{M}, v \models p$ and as $\mathcal{M}, w \models \Box q$ we get $\mathcal{M}, v \models q$. So $\mathcal{M}, v \models p \land q$. As $v$ is arbitrary we get that $\mathcal{M}, w \models \Box(p \land q)$. So $\mathcal{M}, w \models (\Box p \land \Box q) \rightarrow (\Box p \land q)$. As $\mathcal{M}$ and $w$ are arbitrary we get that $\models (\Box p \land \Box q) \leftrightarrow (\Box p \land \Box q)$.

5. $(\neg \Box p \land \neg \Box \neg p)$ is not valid. Consider the following model $\mathcal{M}'$:

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\begin{tikzpicture}
  \node (w0) at (0,0) {$w_0$};
  \node (w1) at (1,0) {$w_1$};
  \node (w2) at (2,0) {$w_2$};
  \node (p) at (1,1) {$p$};
  \draw[->] (w0) -- (w1);
  \draw[->] (w1) -- (p);
  \draw[->] (p) -- (w2);
\end{tikzpicture}
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We get that $\mathcal{M}', w_0 \models \neg \Box p$. This is the case as the only successor of $w_0$ is $w_1$ and $\mathcal{M}', w_1 \models p$, hence $\mathcal{M}', w_0 \not\models \Box p$. But $(\neg \Box p \land \neg \Box \neg p)$ is satisfiable as we have that $\mathcal{M}', w_1 \not\models \neg \Box p \land \neg \Box \neg p$. As it does not have any successor we directly get that $\mathcal{M}', w_1 \not\models \Box p$ and $\mathcal{M}', w_1 \not\models \Box \neg p$, hence $\mathcal{M}', w_1 \not\models \Box(p \land \neg p)$. We thus get $\mathcal{M}', w_1 \not\models \neg \Box p \land \neg \Box \neg p$.

6. $(\Box p \land \Box \neg p) \lor \Diamond T$ is valid. Let $\mathcal{M} = (W, R, V)$ and $w \in W$. If there is a $v \in W$ such that $w R v$ then $\mathcal{M}, v \models T$ by definition and thus $\mathcal{M}, w \models \Diamond T$, hence $\mathcal{M}, w \models (\Box p \land \Box \neg p) \lor \Diamond T$. If there is no such $v$ then we get that $\mathcal{M}, w \models \Box p$ and $\mathcal{M}, w \models \Box \neg p$ and thus $\mathcal{M}, w \models (\Box p \land \Box \neg p)$ hence $\mathcal{M}, w \models (\Box p \land \Box \neg p) \lor \Diamond T$. In both cases we get that $\mathcal{M}, w \models (\Box p \land \Box \neg p) \lor \Diamond T$. As $\mathcal{M}$ and $w$ are arbitrary we get that $\models (\Box p \land \Box \neg p) \lor \Diamond T$.

**Exercise 3.** Let $\mathcal{M} = (W, R, V)$ be a Kripke model and $\phi$ a formula. Prove the following: if $\mathcal{M} \models \phi$, then for every $w \in W$ we have $\mathcal{M}, w \models \Diamond \neg \phi$ or $\mathcal{M}, w \models \Diamond \phi$.

**Answer.** Assume that $\mathcal{M} \models \phi$. Let $w \in W$. By $\mathcal{M} \models \phi$ we get that $\mathcal{M}, w \models \phi$. If there is no $v \in W$ such that $w R v$ then we get that $\mathcal{M}, w \models \Diamond \neg \phi$. If there is a $v \in W$ such that $w R v$ then we claim that $\mathcal{M}, v \models \Box \phi$. Let $u \in W$ such that $v R u$. Then by $\mathcal{M} \models \phi$ we get $\mathcal{M}, u \models \phi$. As $u$ is arbitrary we get that $\mathcal{M}, v \models \Box \phi$. As $w$ was arbitrary we get that for every $w \in W$ we have $\mathcal{M}, w \models \Diamond \neg \phi$ or $\mathcal{M}, w \models \Diamond \phi$.

**Exercise 4.** Show the following:

1. $\Box p \rightarrow p \models \neg p \rightarrow \Diamond \neg p$
2. $\Box p \rightarrow p, \Diamond \neg p \rightarrow \Box p \models \neg p \rightarrow \Diamond \neg p$
3. \( \Box p \to p \models \Diamond \Box p \to \Diamond p \)

Answer.

1. Let \( \mathcal{M} = (W, R, V) \) be a model, and assume that \( \mathcal{M} \models \Box p \to p \). We want to show that \( \mathcal{M} \models \neg \Box \neg p \to \Diamond \neg p \). Let \( w \in W \). Assume that \( \mathcal{M}, w \models \neg \Box p \). Then we have that \( \mathcal{M}, w \not
p \). As \( \mathcal{M} \models \Box p \to p \) we get \( \mathcal{M}, w \models \Diamond \Box p \to p \). But as \( \mathcal{M}, w \not
p \) we can deduce that \( \mathcal{M}, w \not
\Diamond p \) (otherwise we reach a contradiction).

By definition of the semantics we get that there is a \( v \in W \) such that \( wRv \) and \( \mathcal{M}, v \not
p \). So there is a \( v \in W \) such that \( wRv \) and \( \mathcal{M}, v \not
p \). Thus \( \mathcal{M}, w \not
\Diamond \neg p \). Consequently \( \mathcal{M}, w \models \neg \neg p \to \Diamond \neg p \). As \( w \) is arbitrary we get that \( \mathcal{M} \models \neg \neg p \to \Diamond \neg p \). As \( \mathcal{M} \) is arbitrary we get that \( \Diamond \Box p \to p \models \neg p \to \Diamond p \).

2. Let \( \mathcal{M} = (W, R, V) \) be a model, and assume that \( \mathcal{M} \models \Box p \to p \) and \( \mathcal{M} \models \Diamond \Diamond p \to \Box p \). We want to show that \( \mathcal{M} \models \neg \Box \neg p \to \Diamond \neg p \). Let \( w \in W \). Assume that \( \mathcal{M}, w \models \neg p \), hence \( \mathcal{M}, w \not
p \). As \( \mathcal{M} \models \Box p \to p \) we get \( \mathcal{M}, w \models \Box p \to p \). But as \( \mathcal{M}, w \not
p \) we can deduce that \( \mathcal{M}, w \not
\Diamond p \). And as \( \mathcal{M} \models \Box p \to p \) we get that \( \mathcal{M}, w \not
\Diamond \Diamond p \to \Box p \). But as \( \mathcal{M}, w \not
\Diamond p \) we can deduce that \( \mathcal{M}, w \not
\Diamond \Diamond p \). By definition of the semantics we have that for every \( v \in W \) such that \( wRv \), \( \mathcal{M}, v \not
\Diamond p \). This is equivalent to the following: for every \( v \in W \) such that \( wRv \), there is a \( u \in W \) such that \( vRu \) and \( \mathcal{M}, u \not
p \), i.e. \( \mathcal{M}, u \models \neg p \). So for every \( v \in W \) such that \( wRv \), \( \mathcal{M}, v \models \neg \neg p \). Thus \( \mathcal{M}, w \models \Diamond \Diamond \neg p \). Consequently \( \mathcal{M}, w \models \neg p \to \Diamond \Diamond \neg p \). As \( w \) is arbitrary we obtain \( \mathcal{M} \models \neg \Box \neg p \to \Diamond \neg p \). And as \( \mathcal{M} \) is arbitrary we get that \( \Diamond \Box p \to p \models \neg p \to \Diamond \Diamond \neg p \).

Exercise 5. Let \( \Gamma \) be a set of formulae. Show the following:

1. the canonical model \( \mathcal{M}_\Gamma \) based on the logic \( \mathbf{K}4 \) is transitive.

2. the canonical model \( \mathcal{M}_\Gamma \) based on the logic \( \mathbf{K}2 \) is weakly directed (HARD).

3. the canonical model \( \mathcal{M}_\Gamma \) based on the logic \( \mathbf{K}+\Box(\phi \to \Diamond \phi) \) is one-step reflexive, i.e. satisfies \( \forall w, v \in W_c (wRv \to vRv) \).

Answer.

1. Let \( \mathcal{M}_\Gamma = (W_c, R_c, V_c) \) be the canonical model for \( \Gamma \) based on the logic \( \mathbf{K}4 \).

We need to show that for every \( \Delta_1, \Delta_2, \Delta_3 \in W_c \), if \( \Delta_1 R_c \Delta_2 \) and \( \Delta_2 R_c \Delta_3 \) then \( \Delta_1 R_c \Delta_3 \). Let \( \Delta_1, \Delta_2, \Delta_3 \in W_c \), and assume that \( \Delta_1 R_c \Delta_2 \) and \( \Delta_2 R_c \Delta_3 \). We need to show that \( \Delta_1 R_c \Delta_3 \), i.e. that for every formula \( \phi \), if \( \Box \phi \in \Delta_1 \) then \( \phi \in \Delta_3 \). Let \( \phi \) be a formula and assume that \( \Box \phi \in \Delta_1 \). As \( \Box \psi \to \Box \Box \psi \) is an axiom of the logic under consideration we get that \( \Box \phi \to \Box \Box \phi \in \Delta_1 \). But we know that MCSs are closed under implication, so as \( \Box \phi \to \Box \Box \phi \in \Delta_1 \) and \( \Box \phi \in \Delta_1 \) we get that \( \Box \Box \phi \in \Delta_1 \). Moreover, as \( \Delta_1 R_c \Delta_2 \) we get by definition of \( R_c \) that \( \Box \phi \in \Delta_2 \). In turn, \( \Delta_2 R_c \Delta_3 \) implies that \( \phi \in \Delta_3 \). So we have that if \( \Box \phi \in \Delta_1 \) then \( \phi \in \Delta_3 \). As \( \phi \) is arbitrary we get that \( \Delta_1 R_c \Delta_3 \). As \( \Delta_1, \Delta_2, \Delta_3 \) are arbitrary we get that \( R_c \) is transitive.

2. (HARD) Let \( \mathcal{M}_\Gamma = (W_c, R_c, V_c) \) be the canonical model for \( \Gamma \) based on the logic \( \mathbf{K}2 \). We need to show that for every \( \Delta_1, \Delta_2, \Delta_3 \in W_c \), if \( \Delta_1 R_c \Delta_2 \) and \( \Delta_1 R_c \Delta_3 \)
then there is a $\Delta_1 \in W_c$ such that $\Delta_2 R_c \Delta_4$ and $\Delta_3 R_c \Delta_4$. Let $\Delta_1, \Delta_2, \Delta_3 \in W_c$ and assume that $\Delta_1 R_c \Delta_2$ and $\Delta_1 R_c \Delta_3$. Note that if we prove that $S = \{ \phi \mid \Box \phi \in \Delta_2 \cup \Delta_3 \}$ is consistent, then we can extend it to a MCS $\Delta$ containing $\Gamma^*$ (as for every $\gamma \in \Gamma$, we have that $\Box \gamma \in \Delta_1$, essentially because $\Box \gamma \in \Gamma^*$ and $\Gamma^* \subseteq \Delta_i$ for $i \in \{2, 3\}$) is such that $\Delta_2 R_c \Delta$ and $\Delta_3 R_c \Delta$ as $S \subseteq \Delta$. So let us show that $S$ is consistent. Assume for reductio that $S$ is not consistent. Then there are $\phi_1, \ldots, \phi_n \in S$ such that $\vdash (\phi_1 \land \ldots \land \phi_n) \rightarrow \bot$. In fact we can assume that these formulas $\phi_1, \ldots, \phi_n$ can be divided in $\psi_1, \ldots, \psi_l$ such that $\Box \psi_1, \ldots, \Box \psi_l \in \Delta_2$ and $\chi_t, \ldots, \chi_n$ such that $\Box \chi_t, \ldots, \Box \chi_n \in \Delta_3$ (assuming that $\phi_1, \ldots, \phi_n$ are all in either $\Delta_2$ or $\Delta_3$ leads straightforwardly to a contradiction). So we have $\vdash (\psi_1 \land \ldots \land \psi_l \land \chi_t \land \ldots \land \chi_n) \rightarrow \bot$. We can deduce from this that $\vdash (\psi_1 \land \ldots \land \psi_l) \rightarrow \neg(\chi_t \land \ldots \land \chi_n)$ by CPL. Then we easily obtain by necessitation and then the axiom $K$ that $\vdash \Box (\psi_1 \land \ldots \land \psi_l) \rightarrow \Box \neg(\chi_t \land \ldots \land \chi_n)$, i.e. $\vdash \Box (\psi_1 \land \ldots \land \psi_l) \rightarrow \neg \Box (\chi_t \land \ldots \land \chi_n)$. As we know that $\vdash (\Box \psi_1 \land \ldots \land \Box \psi_l) \rightarrow (\psi_1 \land \ldots \land \psi_l)$ holds in modal logic, we get that $\vdash (\Box \psi_1 \land \ldots \land \Box \psi_l) \rightarrow \Box \neg (\chi_t \land \ldots \land \chi_n)$. But then we also know that if $\vdash \rho_1 \rightarrow \rho_2$ then $\vdash \Box \rho_1 \rightarrow \Box \rho_2$ holds in modal logic, so we can obtain $\vdash \Box (\psi_1 \land \ldots \land \Box \psi_l) \rightarrow \neg \Box (\chi_t \land \ldots \land \chi_n)$, i.e. $\vdash \Box (\psi_1 \land \ldots \land \Box \psi_l) \rightarrow \neg \Box (\chi_t \land \ldots \land \chi_n)$. 

Now we prove two things leading to a contradiction:

- $\neg \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$: we know that $\Box \psi_1, \ldots, \Box \psi_l \in \Delta_2$ as it is a MCS (they are closed under conjunction) and $\Box \psi_1, \ldots, \Box \psi_l \in \Delta_2$. By Lemma 10 of the lecture notes (case for $\Box$) we get that $\Box (\Box \psi_1 \land \ldots \land \Box \psi_l) \in \Delta_1$ as $\Delta_1 R_c \Delta_2$. But then as $\Delta_1$ is a MCS and thus closed under implication, and $\vdash (\Box \psi_1 \land \ldots \land \Box \psi_l) \rightarrow \neg \Box (\chi_t \land \ldots \land \chi_n)$, we get that $\neg \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$.

- $\Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$: We also know that $\Box \chi_t, \ldots, \Box \chi_n \in \Delta_3$, hence $\Box \chi_t, \ldots, \Box \chi_n \in \Delta_3$, which implies that $\Box (\chi_t \land \ldots \land \chi_n) \in \Delta_3$. But then by Lemma 10 of the lecture notes again we get that $\Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$, as $\Delta_1 R_c \Delta_3$. Finally we know that $\Box \Box \rho \rightarrow \Box \rho$ is an axiom of the logic here, so we obtain as an instance $\vdash \Box \Box (\chi_t \land \ldots \land \chi_n) \rightarrow \Box (\chi_t \land \ldots \land \chi_n)$. So we have that $\Box \Box (\chi_t \land \ldots \land \chi_n) \rightarrow \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$ and $\Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$ so $\Box \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$.

We then reached our contradiction: we managed to obtain both $\Box \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$ and $\neg \Box (\chi_t \land \ldots \land \chi_n) \in \Delta_1$ which is in contradiction with the consistency of $\Delta_1$. Consequently our assumption according to which $S$ is not consistent is contradictory. Then $S$ is consistent. So by the Lindenbaum lemma it can be extended to a MCS $\Delta$ that is an extension of $\Gamma^*$ (as all the $\gamma \in \Gamma^*$ are such that $\gamma \in S$). As $\Delta$ is an extension of $S$, which implies by construction of $S$ that for every formula $\phi$, if $\Box \phi \in \Delta_2$ or $\Box \phi \in \Delta_3$ then $\phi \in \Delta$, then we get that $\Delta_2 R_c \Delta$ and $\Delta_3 R_c \Delta$. Thus there is a $\Delta$ such that $\Delta_2 R_c \Delta$ and $\Delta_3 R_c \Delta$. Then, as $\Delta_1, \Delta_2, \Delta_3$ are arbitrary, we get that $R_c$ is weakly directed.

3. Let $M_T = (W_c, R_c, V_c)$ be the canonical model for $\Gamma$ based on the logic $K + \Box (p \rightarrow \Box p)$. We need to show that for every $\Delta_1, \Delta_2 \in W_c$, if $\Delta_1 R_c \Delta_2$ then $\Delta_2 R_c \Delta_1$. Let $\Delta_1, \Delta_2 \in W_c$. Assume that $\Delta_1 R_c \Delta_2$. We need to show that $\Delta_2 R_c \Delta_1$. Let $\phi$ be a formula and assume that $\Box \phi \in \Delta_2$. We need to show that $\phi \in \Delta_1$. Assume for reductio that $\phi \notin \Delta_2$. But then by maximality we get that $\neg \phi \in \Delta_2$. As we have the axiom $\Box (\phi \rightarrow \Box \phi)$ we can easily get that $\Box (\neg \phi \rightarrow \Box \neg \phi) \in \Delta_1$. But then by definition of $R_c$ we get that $\neg \phi \rightarrow \Box \neg \phi \in \Delta_2$. And as $\neg \phi \in \Delta_2$ by assumption and MCSs are closed under implication, then we get $\Box \neg \phi \in \Delta_2$. From this we easily deduce that $\Box \neg \phi \in \Delta_2$, which is in contradiction with our assumption that $\Box \phi \in \Delta_2$. So we have that $\phi \in \Delta_1$. As $\phi$ is arbitrary we get that $\Delta_2 R_c \Delta_1$. And as $\Delta_1, \Delta_2$ are arbitrary we get that $R_c$ is one-step reflexive.