 Modal and Temporal Logics - Answers to Exercises 2

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Exercise 1. Prove that a frame:

- satisfies $\forall w \forall v (wRv \rightarrow vRw)$ iff it validates $p \rightarrow \lozenge \lozenge p$;
- satisfies $\forall w \forall v (wRv \rightarrow \forall u (wRu \rightarrow v = u))$ iff it validates $\lozenge p \rightarrow \Box p$;
- satisfies $\forall w \exists v (wRv \land \exists u (wRu \land uRv))$ iff it validates $\Box \Box p \rightarrow \lozenge p$;
- satisfies $\forall w (wRv \land \forall v (wRv \rightarrow w = v))$ iff it validates $(\lozenge p \land q) \rightarrow (p \land \Box q)$;

Answer. Let $F = (W, R)$ be a frame.

First, assume that $F$ satisfies $\forall w \forall v (wRv \rightarrow vRw)$. We show that it validates $p \rightarrow \Box \lozenge p$. Assume for reductio that $F \not\models p \rightarrow \Box \lozenge p$, i.e. there is a valuation $\vartheta$ and a $w \in W$ such that $M, w \not\models p \rightarrow \Box \lozenge p$ where $M = (W, R, \vartheta)$. Then we have that $M, w \models p$ and $M, w \not\models \lozenge p$. From the latter we get that there is a $v \in W$ such that $wRv$ and $M, v \not\models \lozenge p$. However, from the fact that $wRv$ and the property of $F$ we get that $vRw$. And as $M, w \models p$ we get $M, v \models \lozenge p$, hence a contradiction. Consequently $F \models p \rightarrow \Box \lozenge p$.

Second, assume that $F \models p \rightarrow \Box \lozenge p$. We show that $F$ satisfies $\forall w \forall v (wRv \rightarrow vRw)$. Let $w, v \in W$. Assume that $wRv$. We want to show that $vRw$. Assume for reductio that $vRw$ does not hold. Now consider a valuation $\vartheta$ such that $\vartheta(x, p) = t$ iff $x = w$. Then we know that $M, w \models p$, where $M = (W, R, \vartheta)$. Also, as $F \models p \rightarrow \Box \lozenge p$ we get that $M, w \models p \rightarrow \Box \lozenge p$. Consequently, we know that $M, w \models \lozenge p$. As $wRv$ we get that $M, v \models \lozenge p$. However, as $\vartheta$ is such that $\vartheta(x, p) = t$ iff $x = w$, and as $\Box \lozenge p$ does not hold, we also have $M, v \not\models \lozenge p$. But this is a contradiction, so $vRw$. As $w$ and $v$ are arbitrary we get that $F$ satisfies $\forall w \forall v (wRv \rightarrow vRw)$.

First, assume that $F$ satisfies $\forall w \forall v (wRv \rightarrow \forall u (wRu \rightarrow v = u))$. We show that $F \models \Box p \rightarrow \Box \Box p$. Let $\vartheta$ a valuation on $F$ and $w \in W$. Assume that $M, w \models \Box p$, where $M = (W, R, \vartheta)$. From this fact we get that there is a $v \in W$ such $wRv$ and $M, v \models p$. Now let $u \in W$ be such that $wRu$. By $wRu$ and the property of $F$, we get that $u = v$. And as $M, v \models p$ we obtain $M, u \models p$. But $u$ is an arbitrary successor of $w$, so we get that $M, w \models \lozenge p$. Consequently $M, w \models \Box \lozenge p \rightarrow \Box p$. As $w$ and $\vartheta$ are arbitrary we get $F \models \Box p \rightarrow \Box \Box p$.

Second, assume that $F \models \Box p \rightarrow \Box \Box p$. We show that $F$ satisfies $\forall w \forall v (wRv \rightarrow \forall u (wRu \rightarrow v = u))$. Let $w, v \in W$. Assume that $wRv$. We want to show that $\forall u (wRu \rightarrow v = u)$. Let $u \in W$ be such that $wRu$. Assume for reductio that $u \neq v$. Let $\vartheta$ be a valuation on $F$ such that $\vartheta(x, p) = t$ iff $x = v$. Thus, from the fact that $wRv$ we get that $M, w \models \Box p$, where $M = (W, R, \vartheta)$. As $F \models \Box p \rightarrow \Box \Box p$ we obtain $M, w \models \Box \lozenge p \rightarrow \Box p$. But $M, w \models \Box \lozenge p$, so $M, w \not\models \Box p$. Also, we have that $wRu$ so we get $M, u \not\models p$. However, we have that $u \neq v$ which implies that $M, u \not\models p$, as $\vartheta$ is such that $\vartheta(x, p) = t$ iff $x = v$. We are thus in presence of a contradiction. Consequently we obtain $u = v$. In turn, as $w, v$ are arbitrary we can deduce that $F$ satisfies $\forall w \forall v (wRv \rightarrow \forall u (wRu \rightarrow v = u))$. 

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Exercise 2. Prove that the formula:

- $\Diamond \top \lor \Box \bot$ is valid by providing a $K$-tableau;
- $\Diamond (p \lor \Box \bot)$ is valid by providing a $KT$-tableau;
- $\lnot \Diamond \Box \bot$ is valid by providing a $K$-tableau;
- $\Diamond \Diamond p \rightarrow \Diamond p$ is valid by providing a $K4$-tableau;
• $\Diamond p \rightarrow \Diamond \Diamond \Diamond p$ is valid by providing a $KT_4$-tableau.

Answer.

• $\Diamond T \lor \Box \bot$: Note that $\top$ and $\bot$ can respectively be defined as $p \lor \neg p$ and $p \land \neg p$.

\[
\begin{align*}
\Box (p \land \neg p) & \land (p \lor \neg p) \\
\Box (p \land \neg p) ; (p \lor \neg p) & (\Diamond K) \\
p \land \neg p ; p \lor \neg p & (\land) \\
p ; \neg p ; p \lor \neg p & (Id)
\end{align*}
\]

• $\Diamond (p \lor \neg p)$:

\[
\begin{align*}
\Box (\neg p \land p) & (T) \\
\neg p \land p ; (\Box (\neg p \land p))^* & (\land) \\
\neg p ; (\Box (\neg p \land p))^* & (Id)
\end{align*}
\]

• $\neg \Box \Box \Box \bot$:

\[
\begin{align*}
\Box \Box \Box (p \land \neg p) & (\Diamond KT) \\
\Box (p \land \neg p) ; (\Box \Box \Box (p \land \neg p))^* & (T) \\
p \land \neg p ; (\Box (p \land \neg p))^* ; (\Box \Box \Box (p \land \neg p))^* & (\land) \\
p ; \neg p ; (\Box (p \land \neg p))^* ; (\Box \Box \Box (p \land \neg p))^* & (Id)
\end{align*}
\]

• $\Diamond \Diamond \Diamond p \rightarrow \Diamond p$:

\[
\begin{align*}
\Diamond \Diamond \Diamond p \land \Box \neg p & (\land) \\
\Diamond \Diamond p ; \Box \neg p & (\Diamond K4) \\
\neg p ; \Diamond p ; \Box \neg p & (\Diamond K4) \\
\neg p ; \Box \neg p & (Id)
\end{align*}
\]

• $\Diamond p \rightarrow \Diamond \Diamond \Diamond p$:

\[
\begin{align*}
\Diamond \Diamond \diamond p \land \Box \neg p & (\land) \\
\Diamond p ; \Box \Diamond \Diamond \Diamond \Diamond \Diamond p & (T) \\
\Box \neg p ; \Diamond p ; (\Box \Diamond \Diamond \Diamond \Diamond \Diamond p)^* & (T) \\
\neg p ; (\Box (\neg p))^* ; \Diamond p ; (\Box \Diamond \Diamond \Diamond \Diamond \Diamond p)^* & (T) \\
\Box \neg p ; \Box \Diamond \Diamond \Diamond \Diamond \Diamond p ; p \Diamond \Diamond \Diamond \Diamond \Diamond p & (\Diamond KT4) \\
\neg p ; (\Box (\neg p))^* ; \Box \Diamond \Diamond \Diamond \Diamond \Diamond p ; p \Diamond \Diamond \Diamond \Diamond \Diamond p & (T) \\
\neg p ; (\Box (\neg p))^* ; \Box \Diamond \Diamond \Diamond \Diamond \Diamond p ; p \Diamond \Diamond \Diamond \Diamond \Diamond p & (Id)
\end{align*}
\]

Exercise 3. Determine if:

• $\vdash p_0 \rightarrow \Box p_0$ holds using the $K$-tableau calculus;

• $\{p_0\} \models \Box p_0$ holds using the $K$-tableau calculus;
• $\models \Diamond p_0 \rightarrow \Diamond \Diamond p_0$ holds using the $K$-tableau calculus;
• $\{\Diamond p_0\} \vdash \Diamond \Diamond p_0$ holds using the $K$-tableau calculus.

Answer.

• We claim that $\not\models p_0 \rightarrow \Box p_0$. Consider the following tableau:

\[
\begin{array}{c}
p_0 \land \neg p_0 \\
\hline
p_0 \\
\hline
\neg p_0
\end{array}
\]  
\[
(\land) \\
(\Box K)
\]

It is the only tableau we can create for the formula $p_0 \land \neg p_0$ and it is not closed. Then, by completeness of the $K$-tableau calculus with respect to the Kripke semantics on the class of all frames, we get that $p_0 \land \neg p_0$ is $K$-satisfiable. Thus, $\not\models p_0 \rightarrow \Box p_0$. By soundness of the Hilbert style system for the modal logic $K$ with respect to the Kripke semantics on the class of all frames, we get that $\not\models p_0 \rightarrow \Box p_0$.

• We claim that $\{p_0\} \models \Box p_0$. Consider the following tableau:

\[
\begin{array}{c}
p_0; \neg p_0 \\
\hline
\neg p_0
\end{array}
\]  
\[
(\Diamond \Gamma K)
\]

Thus $p_0 \vdash \Box p_0$, and by soundness we obtain $p_0 \models \Box p_0$.

• We claim that $\not\models \Diamond p_0 \rightarrow \Diamond \Diamond p_0$. Consider the following tableau:

\[
\begin{array}{c}
\Diamond p_0 \land \Box \Box \neg p_0 \\
\hline
\Diamond p_0; \Box \Box \neg p_0 \\
\hline
\Box \neg p_0
\end{array}
\]  
\[
(\land) \\
(\Diamond K)
\]

It is the only tableau for $\Diamond p_0 \land \Box \Box \neg p_0$ and it is not closed. Then, by completeness we get that $\not\models \Diamond p_0 \rightarrow \Diamond \Diamond p_0$.

• We claim that $\{\Diamond p_0\} \vdash \Diamond \Diamond p_0$. Consider the following tableau:

\[
\begin{array}{c}
\Diamond p_0; \Box \neg p_0 \\
\hline
\Diamond \neg p_0
\end{array}
\]  
\[
(\Diamond \Gamma K)
\]

Thus we have $\Diamond p_0 \vdash \Diamond \Diamond p_0$, which, by soundness, implies that $\Diamond p_0 \models \Diamond \Diamond p_0$. By completeness of the Hilbert style system for $K$ we obtain $\Diamond p_0 \vdash \Diamond \Diamond p_0$.

Exercise 4. Determine the following:

• does $\Diamond \top$ have a closed $K$-tableau? If so provide it. If not, show that $\neg \Diamond \top$ is $K$-satisfiable.

• does $\Diamond \Box p$ have a closed $KT4$-tableau? If so provide it. If not, show that $\neg (\Diamond \Box p \rightarrow \Diamond p)$ is $KT4$-satisfiable.

• does $\Box p \rightarrow \Diamond p$ have a closed $K4$-tableau? If so provide it. If not, show that $\neg (\Box p \rightarrow \Diamond p)$ is $K4$-satisfiable.
• does ◦(p ∧ q) → (◦p ∧ ◦q) have a closed K-tableau? If so provide it. If not, show that ¬(◦(p ∧ q) → (◦p ∧ ◦q)) is K-satisfiable.

• does (□p ∧ ◦q) → ◦(◦□p ∧ ◦q) have a closed KT4-tableau? If so provide it. If not, show that ¬((□p ∧ ◦q) → ◦(◦□p ∧ ◦q)) is KT4-satisfiable.

Answer.

• ◦⊤ does not have a closed K-tableau. The NNF of ¬◦⊤ is □(p ∧ ¬p), and in fact no K-tableau rule is applicable to this multiset! We can show that ¬◦⊤ is K-satisfiable with the following model M = (W = {w}, R = ∅, ϑ): obviously M, w ⊨ ¬◦⊤, i.e. M, w ⊭ ◦⊤, as w has no successor.

• ◦□p → ◦p does have a closed KT4-tableau:

\[
\begin{align*}
\text{(A)} & \quad ◦□p \land □¬p \\
\text{(T)} & \quad ◦□p; □¬p \\
\text{(KT4)} & \quad ¬p; ◦□p; (□¬p)^* \\
\text{(T)} & \quad ◦□p; □¬p \\
\text{(KT4)} & \quad ¬p; ◦□p; (□¬p)^* \\
\text{(T)} & \quad □p; □¬p \\
\text{(Id)} & \quad p; (□p)^*; □¬p \\
\times & \quad ¬p; p; (□p)^*; (□¬p)^* \\
\end{align*}
\]

• □◦p → ◦p does not have a closed K4-tableau. The NNF of ¬(□◦p → ◦p) is □◦p ∧ □¬p, which has only one tableau:

\[
\begin{align*}
\text{(A)} & \quad □◦p \land □¬p \\
\text{(Id)} & \quad □◦p; □¬p \\
\end{align*}
\]

This tableau indicates that the model M = (W = {w}, R = ∅, ϑ) is such that M, w ⊨ □◦p and M, w ⊭ □¬p. This implies that M, w ⊨ □◦p and M, w ⊬ ◦p, hence M, w ⊬ □◦p → ◦p. Consequently M, w ⊨ ¬(□◦p → ◦p). As M is trivially transitive, we get that ¬(□◦p → ◦p) is K4-satisfiable.

• ◦(p ∧ q) → (◦p ∧ ◦q) does have a closed K-tableau:

\[
\begin{align*}
\text{(A)} & \quad ◦(p \land q) \land (□¬p \lor □¬q) \\
\text{K (V)} & \quad ◦(p \land q); □¬p \\
\text{(Id)} & \quad p \land q; □¬p \\
\end{align*}
\]

• (□p ∧ ◦q) → ◦(◦□p ∧ ◦q) does have a closed KT4-tableau:
Exercise 5. Determine if the following formulae are $K_t$ valid. If so justify your claim. If not provide a countermodel:

- $\phi \to [F] \langle P \rangle \phi$
- $([F] \phi \land [P] \phi) \to \phi$
- $\langle F \rangle [P] (\phi \lor (P) \psi) \to (\neg \phi \to (P) \psi)$
- $[F][P] (\phi \lor \psi) \to ([P][F] \phi \lor [P][F] \psi)$

Answer.

- $\phi \to [F] \langle P \rangle \phi$ is valid. Let $\mathcal{M} = (W, R, \emptyset)$ be a Kripke tense model and $w \in W$. Assume that $\mathcal{M}, w \models \phi$. Let $v \in W$ be such that $wRv$. Then we have $\mathcal{M}, v \models \langle P \rangle \phi$ as $wRv$ and $\mathcal{M}, w \models \phi$. But $v$ is arbitrary, so we obtain $\mathcal{M}, w \models [F] \langle P \rangle \phi$. Thus $\mathcal{M}, w \models \phi \to [F] \langle P \rangle \phi$. As $w$ and $\mathcal{M}$ are arbitrary, we get that $\models \phi \to [F] \langle P \rangle \phi$.

- $([F] \phi \land [P] \phi) \to \phi$ is not valid. Consider the following Kripke tense model $\mathcal{M} = (W = \{w\}, R = \emptyset, \emptyset)$ where $\emptyset(w, p) = f$. We have that $\mathcal{M}, w \models [F] \phi$ and $\mathcal{M}, w \models \phi$ if $w$ has no successor or predecessor, hence $\mathcal{M}, w \models [F] \phi \land [P] \phi$. But we have that $\mathcal{M}, w \not\models \phi$ as $\emptyset(w, p) = f$. Thus $\mathcal{M}, w \not\models ([F] \phi \land [P] \phi) \to \phi$, hence $\not\models ([F] \phi \land [P] \phi) \to \phi$.

- $\langle F \rangle [P] (\phi \lor (P) \psi) \to (\neg \phi \to (P) \psi)$ is valid. Let $\mathcal{M} = (W, R, \emptyset)$ be a Kripke tense model and $w \in W$. Assume that $\mathcal{M}, w \models \langle F \rangle [P] (\phi \lor (P) \psi)$ and $\mathcal{M}, w \models \neg \phi$. As $\mathcal{M}, w \models \langle F \rangle [P] (\phi \lor (P) \psi)$ we get that there is a $v \in W$ such that $wRv$ and $\mathcal{M}, v \models [P] (\phi \lor (P) \psi)$. Now, this last fact combined with $wRv$ gives $\mathcal{M}, w \models (\phi \lor (P) \psi)$, hence $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \models (P) \psi$. By assumption we have that $\mathcal{M}, w \models \neg \phi$, hence $\mathcal{M}, w \not\models \phi$. So it must be the case that $\mathcal{M}, w \not\models (P) \psi$. Thus $\mathcal{M}, w \not\models \langle F \rangle [P] (\phi \lor (P) \psi) \to (\neg \phi \to (P) \psi)$. As $w$ and $\mathcal{M}$ are arbitrary, we get that $\not\models \langle F \rangle [P] (\phi \lor (P) \psi) \to (\neg \phi \to (P) \psi)$.

- $[F][P] (\phi \lor \psi) \to ([P][F] \phi \lor [P][F] \psi)$ is not valid. Consider the following Kripke tense model $\mathcal{M}$:
Exercise 6. Show the following:

- \([P][P] \phi \rightarrow [P] \phi\) is valid on dense (\(\forall w \forall v (vRw \rightarrow \exists u (wRu \land uRv))\) Kripke tense frames;
- \(⟨F⟩[P] \phi \rightarrow [P] \phi\) is valid on transitive Kripke tense frames;
- \([F] \phi \rightarrow ⟨F⟩ \phi\) is valid on Kripke tense frames without right endpoints (\(\forall w \exists v (vRw)\));
- \(\phi \rightarrow ⟨P⟩⟨F⟩ \phi\) is valid on reflexive Kripke tense frames.

\[\]

**Answer.**

- Let \(F = (W, R)\) be a Kripke tense frame. Assume that \(F\) is dense. We show that \(F \models [P][P] \phi \rightarrow [P] \phi\). Let \(\vartheta\) be a valuation on \(F\) and \(w \in W\). Assume that \(M, w \models [P][P] \phi\), where \(M = (W, R, \vartheta)\). Now let \(v \in W\) such that \(vRw\). As \(vRw\) and \(F\) is dense, we obtain that there is a \(u \in W\) such that \(wRu\) and \(uRv\). As \(wRu\) and \(M, w \models [P][P] \phi\) we get that \(M, u \models [P] \phi\). In turn, as \(vRw\) and \(M, u \models [P] \phi\), we obtain \(M, v \models \phi\). As \(v\) is arbitrary we get \(M, w \models [P] \phi\), hence \(M, w \models [P][P] \phi \rightarrow [P] \phi\). Finally, as \(w\) and \(\vartheta\) are arbitrary we get that \(F \models [P][P] \phi \rightarrow [P] \phi\).

- Let \(F = (W, R)\) be a Kripke tense frame. Assume that \(F\) is transitive. We show that \(F \models ⟨F⟩[P] \phi \rightarrow [P] \phi\). Let \(\vartheta\) be a valuation on \(F\) and \(w \in W\). Assume that \(M, w \models ⟨F⟩[P] \phi\), where \(M = (W, R, \vartheta)\). As \(M, w \models ⟨F⟩[P] \phi\) there is a \(u \in W\) such that \(wRu\) and \(M, u \models [P] \phi\). Now let \(v \in W\) such that \(vRw\). As \(vRw\) and \(wRu\) and \(F\) is transitive, we get \(vRu\). From \(vRw\) and \(M, u \models [P] \phi\) we get \(M, v \models \phi\). But \(v\) is arbitrary, so we obtain \(M, w \models [P] \phi\), hence \(M, w \models ⟨F⟩[P] \phi \rightarrow [P] \phi\). Finally, as \(\vartheta\) and \(w\) are arbitrary we have \(F \models ⟨F⟩[P] \phi \rightarrow [P] \phi\).

- Let \(F = (W, R)\) be a Kripke tense frame. Assume that \(F\) has no right endpoints. We show that \(F \models [F] \phi \rightarrow ⟨F⟩ \phi\). Let \(\vartheta\) be a valuation on \(F\) and \(w \in W\). Assume that \(M, w \models [F] \phi\), where \(M = (W, R, \vartheta)\). As \(F\) has no right endpoints we deduce that there is a \(v \in W\) such that \(wRv\). But \(M, w \models [F] \phi\), so \(M, v \models \phi\). Thus \(M, w \models ⟨F⟩ \phi\). Consequently \(M, w \models [F] \phi \rightarrow ⟨F⟩ \phi\). As \(\vartheta\) and \(w\) are arbitrary we get \(F \models [F] \phi \rightarrow ⟨F⟩ \phi\).

- Let \(F = (W, R)\) be a Kripke tense frame. Assume that \(F\) is reflexive. We show that \(F \models \phi \rightarrow ⟨P⟩⟨F⟩ \phi\). Let \(\vartheta\) be a valuation on \(F\) and \(w \in W\). Assume that \(M, w \models \phi\), where \(M = (W, R, \vartheta)\). As \(F\) is reflexive we get \(wRw\). From \(wRw\) and \(M, w \models \phi\) we obtain \(M, w \models ⟨F⟩ \phi\). Moreover, from \(wRw\) and \(M, w \models ⟨F⟩ \phi\) we obtain \(M, w \models ⟨P⟩⟨F⟩ \phi\). Thus \(M, w \models \phi \rightarrow ⟨P⟩⟨F⟩ \phi\). As \(\vartheta\) and \(w\) are arbitrary we get \(F \models \phi \rightarrow ⟨P⟩⟨F⟩ \phi\).